



EUROPEAN CENTRAL BANK

EUROSYSTEM

Working Paper Series

Sulkhan Chavleishvili, Simone Manganelli

Forecasting and stress testing with quantile vector autoregression

Revised December 2020

No 2330 / November 2019

Abstract

A quantile vector autoregressive (VAR) model, unlike standard VAR, models the interaction among the endogenous variables at any quantile. Forecasts of multivariate quantiles are obtained by factorizing the joint distribution in a recursive structure. VAR identification strategies that impose restrictions on the joint distribution can be readily extended to quantile VAR. The model is estimated using real and financial variables for the euro area. The dynamic properties of the system change across quantiles. This is relevant for stress testing exercises, whose goal is to forecast the tail behavior of the economy when hit by large financial and real shocks.

Keywords: Regression quantiles; Multivariate quantiles; Structural VAR; Growth at Risk.

JEL Codes: C32; C53; E17; E32; E44.

NON-TECHNICAL SUMMARY

The standard definition of financial stability adopted by central banks around the world emphasises the negative impact that severe financial shocks may have on real economic activity. This definition underscores an intrinsic tension in connecting the macro and financial dimensions of the economy. The empirical workhorse of macroeconomists is the vector autoregressive (VAR) model, which studies the expected dynamics of the endogenous variables. Financial instability, on the other hand, is inherently linked to the tail dynamics of the system. Using econometric models developed to analyse the average behaviour of macroeconomic variables is bound to miss important aspects of macro-financial linkages which arguably only arise when the system is affected by tail shocks. This paper develops a quantile VAR model, which is designed to address many of the core questions of the macro-finance research agenda.

Quantile VAR models the interaction and feedback effects that the variables of the system have on their quantile dynamics. To study the macro-financial linkages in Europe, we estimate a quantile VAR model on euro area data for industrial production and an indicator of financial distress. We find that financial shocks – defined as a tail quantile realization – are transmitted to the real economy only when the economy is simultaneously hit by a real negative shock. Modelling the mean dynamics with a standard VAR misses most of the action associated with this important channel of transmission of financial shocks. Furthermore, shutting down financial linkages in the system

significantly changes the dynamics of the real economy when hit by negative shocks, but leaves the dynamics largely unaffected in normal conditions. One advantage of quantile VAR is that it allows us to perform impulse response analyses and to forecast the quantiles of the endogenous variables. We find that by hitting the system with a financial shock there is a strong and persistent asymmetric impact on the distribution of industrial production, which takes about two years to be absorbed.

Quantile VAR provides also the natural environment to perform stress testing exercises. To its core, stress testing is a forecast of what happens to the system when it is hit by an arbitrary sequence of negative shocks. If the euro area is hit by a sequence of six monthly consecutive financial and real shocks, its industrial production contracts by a maximum amount of about 4% if the stress scenario were applied in August 2008 and by less than 2% in July 2018. This contrasts with a median forecast of industrial production hovering around 0%.

Finally, this paper contributes to the quantile regression econometric literature by showing how to deal with multiple variables and how to forecast in a time series context. Our econometric framework is general enough to cover the modelling of multiple quantiles of multiple random variables. Stress testing can be thought of as an estimate of the reaction of the endogenous random variables when the system is hit by a sequence of quantile shocks. Stress scenarios are nothing else than an arbitrary series of quantile shocks hitting the macro-financial environment.

1 Introduction

Vector autoregressive (VAR) models are the empirical workhorse of macroeconomics. In their most basic formulation, these models rely on constant coefficients and i.i.d. Gaussian innovations. There is, however, substantial empirical evidence that macroeconomic variables are characterized by nonlinearities and asymmetries which cannot be captured by simple linear Gaussian models (Perez-Quiros and Timmermann 2000, Hubrich and Tetlow 2015, Kilian and Vigfusson 2017, Adrian, Boyarchenko and Giannone 2019). We show how such nonlinearities can be captured by estimating VAR models with quantile regression methods. The insights of our approach can be extended more generally to produce iterated quantile forecasts for nonlinear models. The methodology is applied to the euro area. Using a standard recursive identification scheme, a quantile impulse response analysis reveals that shocks to the financial system have a strong and persistent impact on the left tail of the real economy, but no effect on its right tail.

Quantile regression was introduced by Koenker and Bassett (1978) and has found many applications in economics (Koenker 2005, 2017). Early applications to univariate time series include Engle and Manganelli (2004) and Koenker and Xiao (2006). White, Kim and Manganelli (2010, 2015) develop the asymptotics for multivariate quantile models. In homoskedastic linear regression models, the conditioning variables shift the location of the conditional density of the dependent variables, but they have no effect on

conditional dispersion or shape. In general, however, this needs not be the case. Quantile regression is a semiparametric technique which allows different covariates to affect different parts of the distribution. If and how this happens is an empirical question. In our empirical applications, we find that estimates of quantile regression slopes and quantile impulse response functions vary across quantiles. This may occur either because of time varying higher order moments, and/or because the conditioning variables affect the conditional distribution of the dependent variables in a nonlinear way. These effects cannot be detected with standard OLS VAR estimates.

Quantiles fully describe univariate distributions. A long standing issue in the quantile regression literature, however, is how to deal with multivariate settings. We show how factorizing multivariate distributions into the product between marginal and conditional univariate distributions provides the insight to use quantiles to characterize also the properties of joint distributions. This factorization paves the way to a general framework for quantile forecasting.

For instance, for a bivariate random vector, one can forecast first the quantile of the marginal distribution of the first random variable and then the quantile of the distribution of the second random variable conditional on the first. This reasoning can be repeated recursively for any number of random variables, therefore giving the forecast of the cross section at any given point in time. This intuition holds also for multi step ahead quantile forecasting. The quantile two periods ahead depends on the value taken by

the random variables one period ahead. By conditioning on the quantile values of the random variables one period ahead (which is available from the initial one step ahead forecast just described), we can estimate the quantiles of quantiles of the two step ahead random variables. Iterating this reasoning forward, we can obtain any multi step ahead quantile forecast.

One important issue is identification. In general, the factorization strategy advocated in this paper allows one only to characterize the joint distribution of the random variables. Identification relies on a series of economic restrictions on this joint distribution. If there are economic reasons to follow a particular ordering in the factorization, the model can be given a structural interpretation. In the context of VAR models, this corresponds to a standard Cholesky decomposition. More general identification strategies imposing assumptions on the dynamic properties of first and second moments of the endogenous random variables can be readily extended to our multivariate quantile framework.

One advantage of the multivariate approach is the flexibility to assess the impact of any future quantile realization. Stress testing can be thought of as an estimate of the reaction of the endogenous random variables when the system is hit by a sequence of tail shocks, where tail shocks are defined as future realizations of the random variables being equal to low or high quantile probabilities. Stress scenarios are therefore defined as an arbitrary series (to be chosen by the policy maker or calibrated to past crises) of future quantile realizations hitting the system.

We estimate a quantile VAR model on euro area data for industrial production growth and an indicator of financial distress and perform three types of exercises. First, we estimate euro area growth at risk, defined as the 10% quantile of industrial production growth. We find that severe financial shocks have an asymmetric impact on the distribution of the real variable. Modeling the conditional mean with a standard VAR seriously underestimates these macro-financial dynamics in times of stress, and underscores the potential of quantile VAR models for financial stability purposes. These results are broadly in line with those found by Adrian et al. (2019) for the U.S. economy. The empirical model estimated by Adrian et al. (2019) is equivalent to estimating only one equation of our quantile VAR model. Estimating the full quantile VAR allows us to perform impulse response analyses. We find that by hitting the system with a financial shock there is a strong, persistent and asymmetric impact on the distribution of industrial production, which takes about two years to be absorbed.

Second, we forecast euro area growth under alternative stress scenarios. Quantile VAR provides the natural environment to perform stress testing exercises. At its core, stress testing is a forecast of what happens to the system when hit by an arbitrary sequence of negative shocks. If the euro area is hit by a sequence of six monthly consecutive financial and real 10% quantile realizations, its industrial production contracts by a maximum amount of about 4% if the stress scenario were applied in August 2008 and by less than 2% in July 2018. This contrasts with a median forecast (that is, a sequence

of median realizations of the endogenous variables) of industrial production hovering around 0%.

Third, we perform a counterfactual scenario analysis before Lehman Brothers' default and replay this scenario at each point in time. Using estimates up to August 2008, we find evidence of sizable and unprecedented downside risk to the euro area real economy already in mid 2007. Such counterfactual exercises can help policy makers to better understand the financial stability risks to the economy and put them in an historical perspective.

The paper is organized as follows. Section 2 describes the general framework for quantile forecasting. Section 3 introduces the quantile vector autoregressive model. It provides the links with standard OLS structural VAR and derives the forecasting properties. Section 4 estimates the quantile VAR model for the euro area, performs a stress testing exercise and estimates the counterfactual scenario of Lehman's bankruptcy at each point in time. Section 5 concludes.

2 General framework for forecasting with quantile regression

Quantiles can be used to characterize any part of a univariate distribution. Since any multivariate distribution can be factorized into the product between marginal and conditional univariate distributions, quantiles can be used to characterize also the properties of joint distributions. Exploiting this

intuition, this section presents a general framework for quantile forecasting.

The following assumption characterizes the data generating process.

Assumption 1 (*Data Generating Process*)

1. The observations at time T are $\tilde{y}^T = (\tilde{y}'_0, \tilde{y}'_1, \dots, \tilde{y}'_T)'$, with $\tilde{y}_t \in \mathbb{R}^n$, for $t = 0, 1, \dots, T$.
2. \tilde{y}_t is a realization from the random variable \tilde{Y}_t , an $n \times 1$ vector with i^{th} element denoted by \tilde{Y}_{it} for $i \in \{1, \dots, n\}$.
3. \tilde{Y}_t has pdf $f_t(y_1, \dots, y_n) \equiv f(y_1, \dots, y_n | \tilde{y}^{t-1})$, $t = 1, 2, \dots$, conditional on past observations, with continuous cdf $F_t(y_1, \dots, y_n)$.

Any joint pdf can be decomposed into the product between marginal and conditional densities:

$$f_t(y_1, \dots, y_n) = f_t(y_1) f_t(y_2 | y_1) \dots f_t(y_n | y_1, \dots, y_{n-1})$$

where, with standard notation, the marginal and conditional densities are computed as:

$$f_t(y_1, \dots, y_{i-1}) \equiv \int \dots \int f_t(y_1, \dots, y_n) dy_i \dots dy_n$$

$$f_t(y_i, \dots, y_n | y_1, \dots, y_{i-1}) \equiv f_t(y_1, \dots, y_n) / f_t(y_1, \dots, y_{i-1})$$

Using a simplified notation for the conditional densities:

$$f_t(y_1)f_t(y_2|y_1) \dots f_t(y_n|y_1, \dots, y_{n-1}) \equiv f_{1t}(y_1)f_{2t}(y_2) \dots f_{nt}(y_n) \quad (1)$$

the following theorem shows how to factorize multivariate distributions with conditional quantiles.

Theorem 1 (*Conditional quantile decomposition of cdf*) — Suppose Assumption (1) holds. Let $F_{it}(y_i)$ denote the conditional cdf of $f_{it}(y_i)$ and $q_{it}^{\theta_i} = F_{it}^{-1}(\theta_i)$ the corresponding θ_i quantile, $\theta_i \in (0, 1)$, $i \in \{1, \dots, n\}$, $t = 1, \dots, T$. The joint cdf of $f_t(y_1, \dots, y_n)$ can be decomposed as:

$$\begin{aligned} F_t(q_{1t}^{\theta_1}, \dots, q_{nt}^{\theta_n}) &= F_{1t}(q_{1t}^{\theta_1})F_{2t}(q_{2t}^{\theta_2}) \dots F_{nt}(q_{nt}^{\theta_n}) \\ &= \theta_1\theta_2 \dots \theta_n \end{aligned} \quad (2)$$

Proof — See appendix.

By construction, $q_{it}^{\theta_i}$ is a function of (y_1, \dots, y_{i-1}) for $i > 1$. However, its quantile probability is constant and does not depend on these observations. The corresponding term can therefore be pulled out of the integral, leading to the above independent-like factorization.

The joint distribution at any future time $T + h$, for $h \geq 1$, can be computed as in a simulation procedure (Serfling, 1980). Suppose that, given the information available at time T , one can compute the $T + 1$ conditional quantile forecasts. These quantiles can be used to characterize the joint dis-

tribution at time $T + 1$. Conditional on each quantile at time $T + 1$, one can compute the $T + 2$ conditional quantile forecasts to generate the joint distribution at time $T + 2$. The process can be repeated indefinitely to produce the forecast joint distribution at any future date $T + h$.

Implementation of this procedure requires the choice of a grid of quantile probabilities and a parametric specification for each quantile. The choice of quantile probabilities characterizes the different parts of the density. They should be symmetric around the median, to avoid that specific parts of the density receive disproportional weights. The finer the grid, the more precise the approximation, but the higher the computation cost. For instance, if one chooses p quantiles, the number of branches increases exponentially at each iteration, so that after h periods there are p^{nh} possible ramifications. This curse of dimensionality can be tackled by taking sub-samples.

The quantile parametric specification is needed to compute the forecast associated with each quantile. When multiple quantiles of the same random variable are estimated, a well-known problem is that the monotonicity property of quantile functions can be violated: some estimated quantiles can cross each other. If the quantile model is correctly specified, then the population quantiles are monotonic and quantile crossing is simply a finite sample problem. If the quantile model is misspecified and/or the sample size is not large enough, then quantile crossing can still be of concern. In that case, one can use techniques such as the monotonization method by Chernozhukov et al. (2010), the dynamic additive quantile specification of Gouriéroux and Jasiak

(2008), or the isotonization method suggested by Mammen (1991). Notice, however, that quantile monotonicity (that is, lack of quantile crossing as in a location-scale model) does not imply that the model estimates do not suffer from estimation and/or mis-specification errors.

2.1 Example

Consider the following model from example 3.1 of Engle, Hendry and Richard (1983), to illustrate the logic of Theorem 1:

$$\begin{bmatrix} \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \end{bmatrix} \sim N(\mu, \Sigma) \quad \mu = (\mu_i), \quad \Sigma = (\sigma_{ij}), \quad i, j = 1, 2 \quad (3)$$

Letting $F(y_1, y_2)$ denote the cdf of the bi-variate normal distribution, it can be decomposed as:

$$F(y_1, y_2) = F_1(y_1)F_2(y_2)$$

where $F_1(y_1) \equiv F(y_1; \eta_1, \omega_1)$, $F_2(y_2) \equiv F(y_2; \eta_{2t}, \omega_2)$ and $F(y; \eta, \omega)$ denotes the cdf of the univariate normal distribution with mean η and variance ω . Knowing that $\eta_1 = \mu_1$, $\omega_1 = \sigma_{11}$, $\eta_{2t} = \mu_2 + (\sigma_{12}/\sigma_{11})(y_{1t} - \mu_1)$, and $\omega_2 =$

$\sigma_{22} - \sigma_{12}^2/\sigma_{11}$, the conditional quantiles associated with this model are:

$$q_{1t}^{\theta_1} = \eta_1 + \kappa_{\theta_1} \sqrt{\omega_1}$$

$$q_{2t}^{\theta_2} = \eta_{2t} + \kappa_{\theta_2} \sqrt{\omega_2}$$

where $\theta_i \in (0, 1)$ and κ_θ is the θ -quantile of the standard normal distribution.

Notice that $q_{2t}^{\theta_2}$ depends on y_{1t} via the term η_{2t} . The initial bi-variate normal distribution can therefore be simulated via quantiles by first computing the quantiles for the marginal distribution of y_1 , and then, conditional on each of these quantiles, by computing the quantiles for the conditional distribution of y_2 . This reasoning can be repeated for any multistep ahead forecasting.

2.2 Relationship with identification

As noted by Engle et al. (1983), an equivalent factorization of the bivariate normal model (3) could be obtained by inverting the ordering of the variables. If there are economic reasons to prefer a particular ordering, the model can be given a structural interpretation, as in a standard Cholesky recursive identification. In general, however, the procedure highlighted in this section allows one only to characterize the joint distribution of the random variables of interest.

Identification relies on a series of restrictions on the joint distribution. Since most identification methods in the macro-econometric literature impose

assumptions on the dynamic properties of the first and second moments of the endogenous random variables, and since these moments can be simulated from the joint distribution (Serfling, 1980), it is possible to apply similar identification strategies to the quantile-based model.

The empirical application of this paper appeals to a recursive structural identification. Application of other identification methods is left for future research.

3 Quantile vector autoregression

This section studies the properties of quantile VAR (QVAR). It starts in section 3.1 by introducing the QVAR(1) model, establishes the law of iterated quantiles (section 3.2), and applies the general framework of the previous section to forecasting and stress testing (section 3.3). Section 3.4 generalizes the results to any QVAR(q). Section 3.5 contains details about estimation and asymptotics.

3.1 Quantile VAR(1)

Consider the following vector autoregressive model, written in recursive form:

$$\tilde{Y}_{t+1} = \tilde{\omega} + \tilde{A}_0 \tilde{Y}_{t+1} + \tilde{A}_1 \tilde{Y}_t + \tilde{\epsilon}_{t+1}, \quad \tilde{\epsilon}_{t+1} \sim i.i.d.(0, \Sigma) \quad (4)$$

where \tilde{A}_0 and \tilde{A}_1 are a $n \times n$ coefficient matrices, $\tilde{\omega}$ is a $n \times 1$ vector of constants, $\tilde{\epsilon}_{t+1}$ is a $n \times 1$ vector of i.i.d. structural zero mean shocks with Σ a diagonal matrix. Imposing that \tilde{A}_0 has a lower triangular structure with zeros along the main diagonal is equivalent to factorizing the joint density into the product of marginal and conditional densities, as highlighted in Theorem 1. In the context of the VAR literature, this is also equivalent to identification of the system by assuming a Choleski decomposition of the variance covariance matrix of the residuals from a standard reduced form VAR (see, for instance, chapter 2 of Lütkepohl 2005).

Our goal is to cast model (4) in a quantile regression framework.¹ An explicit example may help to fix concepts, before moving to more general notation. Consider a model with two endogenous random variables and two quantiles to be modeled, say 50% and 90%. A QVAR system can be written

¹See Schüler (2014) for an example of Bayesian quantile structural vector autoregressive model.

explicitly as:

$$\begin{aligned}
\begin{bmatrix} \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \\ \hline \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \end{bmatrix} &= \begin{bmatrix} \omega_1^5 \\ \omega_2^5 \\ \hline \omega_1^9 \\ \omega_2^9 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{021}^5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & a_{021}^9 & 0 \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \\ \hline \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \end{bmatrix} + \quad (5) \\
&+ \begin{bmatrix} a_{11}^5 & a_{12}^5 & 0 & 0 \\ a_{21}^5 & a_{22}^5 & 0 & 0 \\ \hline 0 & 0 & a_{11}^9 & a_{12}^9 \\ 0 & 0 & a_{21}^9 & a_{22}^9 \end{bmatrix} \begin{bmatrix} \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \\ \hline \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t+1}^5 \\ \epsilon_{2,t+1}^5 \\ \hline \epsilon_{1,t+1}^9 \\ \epsilon_{2,t+1}^9 \end{bmatrix}
\end{aligned}$$

The first and second blocks determine, respectively, the dynamics of the 50% and 90% quantiles. The error term are quantile specific and satisfy the condition that $P(\epsilon_{1,t+1}^\theta < 0 | \Omega_t) = \theta$ and $P(\epsilon_{2,t+1}^\theta < 0 | \Omega_t, \tilde{Y}_{1,t+1}) = \theta$, for $\theta \in \{0.50, 0.90\}$, where Ω_t is the information available at time t .

Let us move now to the general setup. Since we want to consider the possibility of jointly modeling multiple quantiles, we need additional notation. For our purposes, it is important to define a recursive information set, which allows us to work with recursive models.

Definition 1 (*Recursive information set*) — *The recursive informa-*

tion set is defined as:

$$\begin{aligned}\Omega_{1t} &\equiv \{\tilde{Y}_t, \tilde{Y}_{t-1}, \dots\} \\ \Omega_{it} &\equiv \{\tilde{Y}_{i-1,t+1}, \Omega_{i-1,t}\} \quad i = 2, \dots, n\end{aligned}$$

According to this definition, the recursive information set Ω_{2t} , say, contains all the lagged values of \tilde{Y}_t as well as the contemporaneous value of $\tilde{Y}_{1,t+1}$.

Considering p distinct quantiles, $0 < \theta_1 < \theta_2 < \dots < \theta_p < 1$, the quantile vector autoregressive model is defined as follows:

$$\begin{aligned}Y_{t+1} &= \omega + A_0 Y_{t+1} + A_1 Y_t + \epsilon_{t+1}, \quad P(\epsilon_{i,t+1}^{\theta_j} < 0 | \Omega_{it}) = \theta_j, \quad (6) \\ i &= 1, \dots, n, \quad j = 1, \dots, p\end{aligned}$$

The dependent variable Y_t is now an np -vector, which is obtained as $Y_t = \iota_p \otimes \tilde{Y}_t$, where ι_p is a p -vector of ones, and $\epsilon_t \equiv [\epsilon_{1t}^{\theta_1}, \dots, \epsilon_{nt}^{\theta_1}, \dots, \epsilon_{1t}^{\theta_p}, \dots, \epsilon_{nt}^{\theta_p}]'$. The matrices A_0 and A_1 are block diagonal, to avoid trivial multicollinearity problems. We further impose that the diagonal blocks of A_0 are lower triangular matrices with zeros along their main diagonal, reflecting the recursive structure of the system. The conditioning information set in the probability defining the regression quantile follows the recursive structure as well.

If system (4) is the data generating process, then $\omega = \iota_p \otimes \tilde{\omega} + \kappa^\theta$, where κ^θ is the np -vector containing the θ quantiles of $\tilde{\epsilon}_{t+1}$, $A_0 = I_p \otimes \tilde{A}_0$ and $A_1 = I_p \otimes \tilde{A}_1$. Under this assumption, the VAR and quantile VAR are

characterized by identical dynamics.

3.2 The law of iterated quantiles

Quantile forecasts in a linear model like (6) can be obtained by taking quantiles of quantiles. The forecasting properties of the system are derived in the next subsection. In this subsection, we clarify the logic underlying quantile forecasts.

Define the quantile operator, for convenience:

Definition 2 (Quantile operator) — *The conditional quantile operator $Q_{it}^\theta(\tilde{Y}_{i,t+1})$ of the random variable $\tilde{Y}_{i,t+1}$, given the information set Ω_{it} , is implicitly defined by:*

$$P(Y_{i,t+1} < Q_{it}^\theta(\tilde{Y}_{i,t+1}) | \Omega_{it}) = \theta$$

Let us work with example (5) to illustrate the intuition of quantile forecasting. Consider, for instance, the line corresponding to the 50% quantile of $\tilde{Y}_{2,t+1}$, which is $\tilde{Y}_{2,t+1} = q_{2t}^5 + a_{021}^5 \tilde{Y}_{1,t+1} + \epsilon_{2,t+1}^5$, where $q_{2t}^5 \equiv \omega_2^5 + a_{21}^5 \tilde{Y}_{1t} + a_{22}^5 \tilde{Y}_{2t}$. The 50% quantile of $\tilde{Y}_{2,t+1}$ conditional on the information set $\Omega_{2t} = \{\tilde{Y}_{1,t+1}, \tilde{Y}_t\}$ is:

$$Q_{2t}^5(\tilde{Y}_{2,t+1}) = q_{2t}^5 + a_{021}^5 \tilde{Y}_{1,t+1}$$

because, by the conditional quantile restriction, $Q_{2t}^5(\epsilon_{2,t+1}^5) = 0$. This quan-

tity is still a random variable at time t , because of the term $a_{021}^5 \tilde{Y}_{1,t+1}$. One can choose to take any quantile of this random variable. Let us take the 90% quantile, which according to model (5) is $Q_{1t}^9(\tilde{Y}_{1,t+1}) = q_{1t}^9$, where $q_{1t}^9 \equiv \omega_1^9 + a_{11}^9 \tilde{Y}_{1t} + a_{12}^9 \tilde{Y}_{2t}$. If $a_{021}^5 > 0$, we can now compute the 90% quantile of the 50% quantile of $\tilde{Y}_{2,t+1}$:²

$$Q_{1t}^9(Q_{2t}^5(\tilde{Y}_{2,t+1})) = q_{2t}^5 + a_{021}^5 q_{1t}^9$$

We formalize the intuition of this example in the following theorem.

Theorem 2 (Law of Iterated Quantiles) — Consider model (6) and let $\vartheta \in [\theta_1, \dots, \theta_p]^n$ be an n -vector with typical element denoted by ϑ_i , for $i = 1, \dots, n$. Then:

$$Q_{1t}^{\vartheta_1}(\dots Q_{i-1,t}^{\vartheta_{i-1}}(Q_{it}^{\vartheta_i}(\epsilon_{1,t+1}^{\vartheta_1} + \dots + \epsilon_{i-1,t+1}^{\vartheta_{i-1}} + \epsilon_{i,t+1}^{\vartheta_i}))) = 0 \quad (7)$$

Proof — See appendix.

An important difference with the law of iterated expectations is that the quantile of the sum of random variables is not necessarily equal to the quantile of the quantile of the sum:

$$Q_t^{\vartheta_i}(\epsilon_{1,t+1}^{\vartheta_i} + \epsilon_{2,t+1}^{\vartheta_i}) \neq Q_{1,t}^{\vartheta_i}(Q_{2,t}^{\vartheta_i}(\epsilon_{1,t+1}^{\vartheta_i} + \epsilon_{2,t+1}^{\vartheta_i}))$$

²Notice that if $a_{021}^5 < 0$, we would be actually computing the 10% quantile of the 50% quantile.

where $Q_t^{\vartheta_i}$ represents the ϑ_i -quantile conditional on the standard information set $\Omega_t \equiv \{\tilde{Y}_t, \tilde{Y}_{t-1}, \dots\}$.

The next subsection shows how, in linear models like (6), it is possible to compute the quantile of any future quantile, but there is no closed form solution for the quantile of future random variables at horizons greater than 1, because they depend on the sum of future residuals. If the interest is in these quantiles, they can always be recovered by simulation, following procedures similar to those outlined in Serfling (1980).

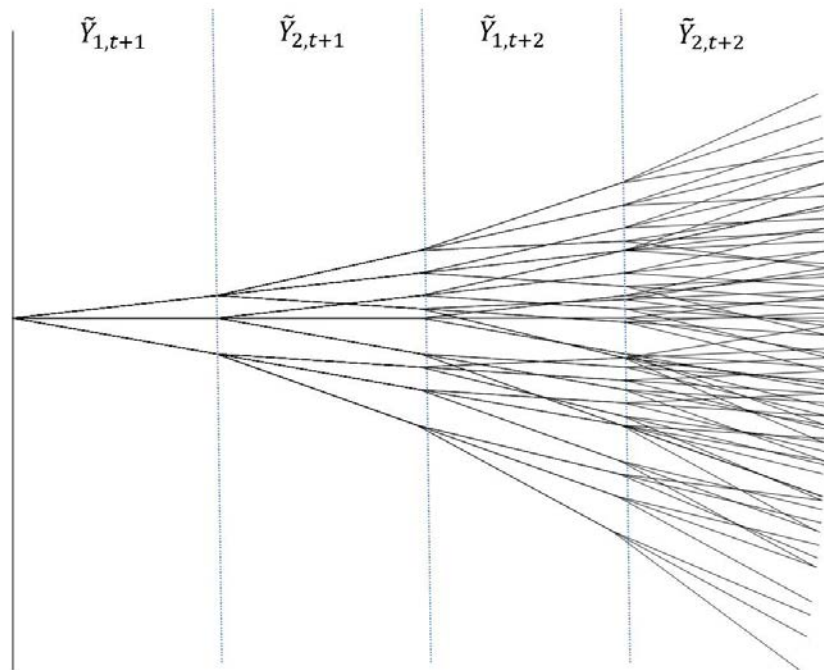
3.3 Forecasting and quantile impulse response functions

This subsection derives the quantile forecast for any combination of future quantiles for model (6).

One can think of quantile forecasts as branches of a tree. An illustrative example is reported in Figure 1, for two variables, two periods ahead and three quantiles, say 10%, 50% and 90%. Exploiting the factorization (2), the starting node, $\tilde{Y}_{1,t+1}$, has three branches (the three quantiles). At the end of each branch, there are three more branches for $\tilde{Y}_{2,t+1}$, corresponding to the one step ahead quantile forecast of the second variable conditional on the quantile forecast of the first variable. The branching continues at $t + 2$, and can go on for any arbitrary number of variables, quantiles and horizon.

Each path in the quantile forecasting tree can be formally identified by

Figure 1: Quantile forecasting tree



Note: Example of possible quantile forecast paths for a model with two variables (\tilde{Y}_1 and \tilde{Y}_2), two periods ahead ($t + 1$ and $t + 2$), where three quantiles are modeled for each variable (the three branches coming out of each node). Increasing the number of quantiles per variable results in a richer branch structure.

defining the following quantile selection matrix.

Definition 3 (Quantile Selection Matrix) — *The **quantile selection matrix** is the $n \times np$ matrix $S_{t+h}^{\vartheta^h}$, for $h \geq 1$, selecting one, and only one, quantile for each endogenous variable from the np -vector ϵ_{t+h} in model (6):*

$$S_{t+h}^{\vartheta^h} \epsilon_{t+h}^\theta = [\epsilon_{1,t+h}^{\vartheta_1^h}, \dots, \epsilon_{n,t+h}^{\vartheta_n^h}]' \quad (8)$$

where $\vartheta^h \in [\theta_1, \dots, \theta_p]^n$ is an n -vector with typical element denoted by ϑ_i^h , for $i = 1, \dots, n$.

Note that $\mathbb{S} \equiv \{S_{t+h}^{\vartheta^h}\}_{h=1}^H$ identifies one entire path in the quantile forecasting tree. The collection of all possible paths is obtained by choosing every different ϑ^h from the set $[\theta_1, \dots, \theta_p]^n$, for all $h = 1, \dots, H$. Since at each period there are p^n possible choices, after H periods there are p^{nH} distinct forecasting paths. In the example of the tree of Figure 1, where $p = 3$, $n = 2$ and $H = 2$, there is a total of 81 distinct paths.

The next theorem derives the generic H step ahead quantile forecasts associated with any path of the quantile forecasting tree identified by \mathbb{S} .

Theorem 3 (Multi step quantile VAR forecast) — *Let $\mathbb{S} \equiv \{S_{t+h}^{\vartheta^h}\}_{h=1}^H$ denote the sequence of quantile selection matrices as in definition 3, selecting the quantiles to be forecasted. The corresponding $n \times 1$ quantile forecasts as of time t associated with process (6) for $H \geq 1$, can be computed recursively*

as:

$$\hat{Y}_{t+1}^{\mathbb{S}} = \bar{B}_{t+1}^{\vartheta^1}(\omega + A_1 Y_t) \quad (9)$$

$$\hat{Y}_{t+h}^{\mathbb{S}} = \bar{B}_{t+h}^{\vartheta^h}(\omega + A_1 \bar{S} \hat{Y}_{t+h-1}^{\mathbb{S}}) \quad \text{for } h \geq 2 \quad (10)$$

where $\bar{B}_{t+h}^{\vartheta^h} \equiv (I_n - S_{t+h}^{\vartheta^h} A_0 \bar{S})^{-1} S_{t+h}^{\vartheta^h}$ for $h = 1, \dots, H$, I_n is the $n \times n$ identity matrix, and \bar{S} is the $np \times n$ duplication matrix stacking p times the n identity matrix.

Proof — See appendix.

To build intuition about the mechanics of quantile forecasting, consider forecasting the sequence of medians. In this case, \mathbb{S} selects H times the median quantile block of system (6). Denoting the respective matrices with $\omega^{\cdot 5}$, $A_0^{\cdot 5}$ and $A_1^{\cdot 5}$ with obvious notation, the median forecast H steps ahead is $\sum_{h=0}^{H-1} B^h \omega + B^H Y_t$, where $\omega \equiv (I_n - A_0^{\cdot 5})^{-1} \omega^{\cdot 5}$ and $B \equiv (I_n - A_0^{\cdot 5})^{-1} A_1^{\cdot 5}$. Notice how this is the median VAR forecast counterpart of the standard mean VAR forecast. Theorem 3 generalizes the forecast to any possible sequence of quantiles.

Unlike the classical VAR, however, the greater generality and flexibility of (9)-(10) provides the natural environment to perform stress testing exercises. A policy maker interested in how the endogenous variables react to a given stress scenario can first define the scenario by choosing a series of future tail quantiles of interest (say, 10%), and then obtain the forecast of the endogenous variables conditional on the chosen scenario.

If the recursive model can be given a structural interpretation, it is possible to derive a structural quantile impulse response function. Express Y_t in terms of structural shocks:

$$Y_t = \nu + BY_{t-1} + (I_{np} - A_0)^{-1}\epsilon_t$$

where $\nu = (I_{np} - A_0)^{-1}\omega$ and $B = (I_{np} - A_0)^{-1}A_1$. In a standard VAR model, a shock to variable i at t is affecting only the conditional expectations. In the case of QVAR, the same shock is affecting all the quantiles. Define the shock to the structural residuals of variable i , for $i = 1, \dots, n$, as follows:

$$\check{\epsilon}_t^i = \epsilon_t + s^i \delta$$

where $\delta \in \mathbb{R}$ and s^i is an np vector of zeros with p ones in the positions corresponding to the quantile residuals of the i^{th} variable. The intuition is that the shock δ is simultaneously applied to all the quantile structural shocks of the i^{th} variable. Denoting with \check{Y}_t the value of the dependent variables if the shock $\check{\epsilon}_t^i$ is applied, the impulse response function at time $t + h$ can then

defined recursively from equations (9)-(10):

$$\begin{aligned}\Delta_t^i &\equiv \ddot{Y}_t - Y_t \\ &= (I_{np} - A_0)^{-1} s^i \delta\end{aligned}\tag{11}$$

$$\Delta_{t+1}^{i\mathbb{S}} = \bar{B}_{t+1}^{\vartheta^1} A_1 \Delta_t^i\tag{12}$$

$$\Delta_{t+h}^{i\mathbb{S}} = \bar{B}_{t+h}^{\vartheta^h} A_1 \bar{S} \Delta_{t+h-1}^{i\mathbb{S}} \quad \text{for } h \geq 2\tag{13}$$

Notice that if one were to model only the median, this is again the median impulse response analogue of the standard mean impulse response function. Quantile impulse response functions, however, will generally depend on the quantiles paths which are considered, and therefore the dependence on the selection matrix \mathbb{S} .

3.4 General quantile VAR(q) model

Model (6) can be generalized to any desired lag order q using its companion form. Define the npq vectors $\bar{\omega} \equiv [\omega', 0', \dots, 0']'$, $\bar{Y}_{t+1} \equiv [Y'_{t+1}, Y'_t, \dots, Y'_{t-q+2}]'$, $\varepsilon_{t+1} \equiv [\epsilon'_{t+1}, 0', \dots, 0']'$, and the $(npq \times npq)$ matrices

$$A^0 = \begin{bmatrix} A_0 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \quad \text{and} \quad A^1 = \begin{bmatrix} A_1 & A_2 & \dots & A_q \\ I_{np} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \dots & I_{np} & \mathbf{0} \end{bmatrix}.$$

Then the companion form of the VAR(q) model is:

$$\bar{Y}_{t+1} = \bar{\omega} + A^0 \bar{Y}_{t+1} + A^1 \bar{Y}_t + \varepsilon_{t+1} \quad (14)$$

All the results of the previous sections extend to model (14).

3.5 Estimation and asymptotics

The recursive QVAR model (6) fits the framework of White et al. (2015), which can therefore be used for inference. Let $q_t(\beta) \equiv \omega + A_0 Y_t + A_1 Y_{t-1}$ and $q_{it}^j(\beta)$ the j^{th} quantile of the i^{th} variable of the vector $q_t(\beta)$, where we have made explicit the dependence on β , the vector containing all the unknown parameters in ω , A_0 , and A_1 . Define the quasi-maximum likelihood estimator $\hat{\beta}$ as the solution of the optimization problem:

$$\hat{\beta} = \arg \min_{\beta} T^{-1} \sum_{t=1}^T \left\{ \sum_{i=1}^n \sum_{j=1}^p \rho_{\theta_j} \left(\tilde{Y}_{it} - q_{it}^j(\beta) \right) \right\}, \quad (15)$$

where $\rho_{\theta}(u) \equiv u(\theta - I(u < 0))$ is the standard check function of quantile regressions. The asymptotic distribution of the regression quantile estimator is provided by White et al. (2015), which we report here for convenience.

Theorem 4 (White et al., 2015) — *Under the assumptions of theorems 1 and 2 of White et al. (2015), $\hat{\beta}$ is consistent and asymptotically normally*

distributed. The asymptotic distribution is:

$$\sqrt{T}(\hat{\beta} - \beta^*) \xrightarrow{d} N(0, Q^{-1}VQ^{-1}) \quad (16)$$

where

$$\begin{aligned} Q &\equiv \sum_{i=1}^n \sum_{j=1}^p E[f_{it}^j(0) \nabla q_{it}^j(\beta^*) \nabla' q_{it}^j(\beta^*)] \\ V &\equiv E[\eta_t \eta_t'] \\ \eta_t &\equiv \sum_{i=1}^n \sum_{j=1}^p \nabla q_{it}^j(\beta^*) \psi^j(\epsilon_{it}^{\theta_j}) \\ \psi^j(\epsilon_{it}^{\theta_j}) &\equiv \theta_j - I(\epsilon_{it}^{\theta_j} \leq 0) \\ \epsilon_{it}^{\theta_j} &\equiv \tilde{Y}_{it} - q_{it}^j(\beta^*) \end{aligned}$$

and $f_{it}^j(0)$ is the conditional density function of $\epsilon_{it}^{\theta_j}$ evaluated at 0.

The asymptotic variance-covariance matrix can be consistently estimated as suggested in theorems 3 and 4 of White et al. (2015), or using bootstrap based methods in the spirit of Buchinsky (1995).³

The following corollary derives the standard errors of the forecasts.

Corollary 1 (Forecast standard errors) — Let $Y_{t+h}(\hat{\beta}) \equiv \hat{Y}_{t+h}^{\mathbb{S}}$ the forecast (9)-(10), where it has been made explicit the dependence on the model

³Modern statistical softwares contain packages for regression quantile estimation and inference. This paper uses the interior point algorithm discussed by Koenker and Park (1996).

parameters β . Then:

$$\sqrt{T}(Y_{t+h}(\hat{\beta}) - Y_{t+h}(\beta^*)) \xrightarrow{d} N(0, \Phi(\beta^*)Q^{-1}VQ^{-1}\Phi'(\beta^*)) \quad (17)$$

where $\Phi(\beta^*) \equiv \partial Y_{t+h}(\beta^*)/\partial \beta'$.

Proof — See appendix.

The standard errors associated with the impulse response functions (11)-(13) can be computed in a similar fashion.

4 Stress testing the euro area economy

We estimate a QVAR(1) to model the interaction between real and financial variables in Europe. We study the interrelationship between the euro area industrial production growth (\tilde{Y}_{1t}) and the composite indicator of systemic stress in the financial system (CISS, \tilde{Y}_{2t}) of Hollo, Kremer and Lo Duca (2012). Our data sample is monthly and ranges from January 1999 to July 2018. We perform three exercises. First, we estimate short term euro area growth at risk (defined as the 10% quantile of \tilde{Y}_{1t}), as a function of financial conditions. Second, we forecast euro area growth under a severe stress scenario, where both the real and financial parts of the euro area economy are hit by a sequence of consecutive tail shocks. Third, we ask whether the quantile VAR methodology could have been helpful in detecting vulnerabilities in the months preceding Lehman Brothers' default.

4.1 Euro area growth at risk

Adrian et al. (2019) have shown that there are substantial asymmetries in the relationship between the US real GDP growth and financial conditions. In particular, they find that the estimated lower quantiles of the distribution of future GDP growth are significantly affected by financial conditions, while the upper quantiles appear to be more stable over time. The quantile model specification of Adrian et al. (2019) is the following:

$$\tilde{Y}_{1,t+1} = \omega_1^\theta + a_{11}^\theta \tilde{Y}_{1,t} + a_{12}^\theta \tilde{Y}_{2t} + \epsilon_{t+1}^\theta \quad (18)$$

They estimate this model for $\theta \in \{0.05, 0.25, 0.75, .95\}$. This corresponds to the first line of model (6). We estimate, instead, the full QVAR model and study its dynamic properties:

$$\tilde{Y}_{1,t+1} = \omega_1^\theta + a_{11}^\theta \tilde{Y}_{1t} + a_{12}^\theta \tilde{Y}_{2t} + \epsilon_{1,t+1}^\theta \quad (19)$$

$$\tilde{Y}_{2,t+1} = \omega_2^\theta + a_{01}^\theta \tilde{Y}_{1,t+1} + a_{21}^\theta \tilde{Y}_{1t} + a_{22}^\theta \tilde{Y}_{2t} + \epsilon_{2,t+1}^\theta \quad (20)$$

By ordering CISS after industrial production, we impose the structural identification assumption that financial variables can react contemporaneously to real variables, but real variables react to financial developments only with a lag. This corresponds to a Choleski identification where shocks to real economic variables can have an immediate impact on financial variables, while shocks to financial variables are allowed to affect real variables

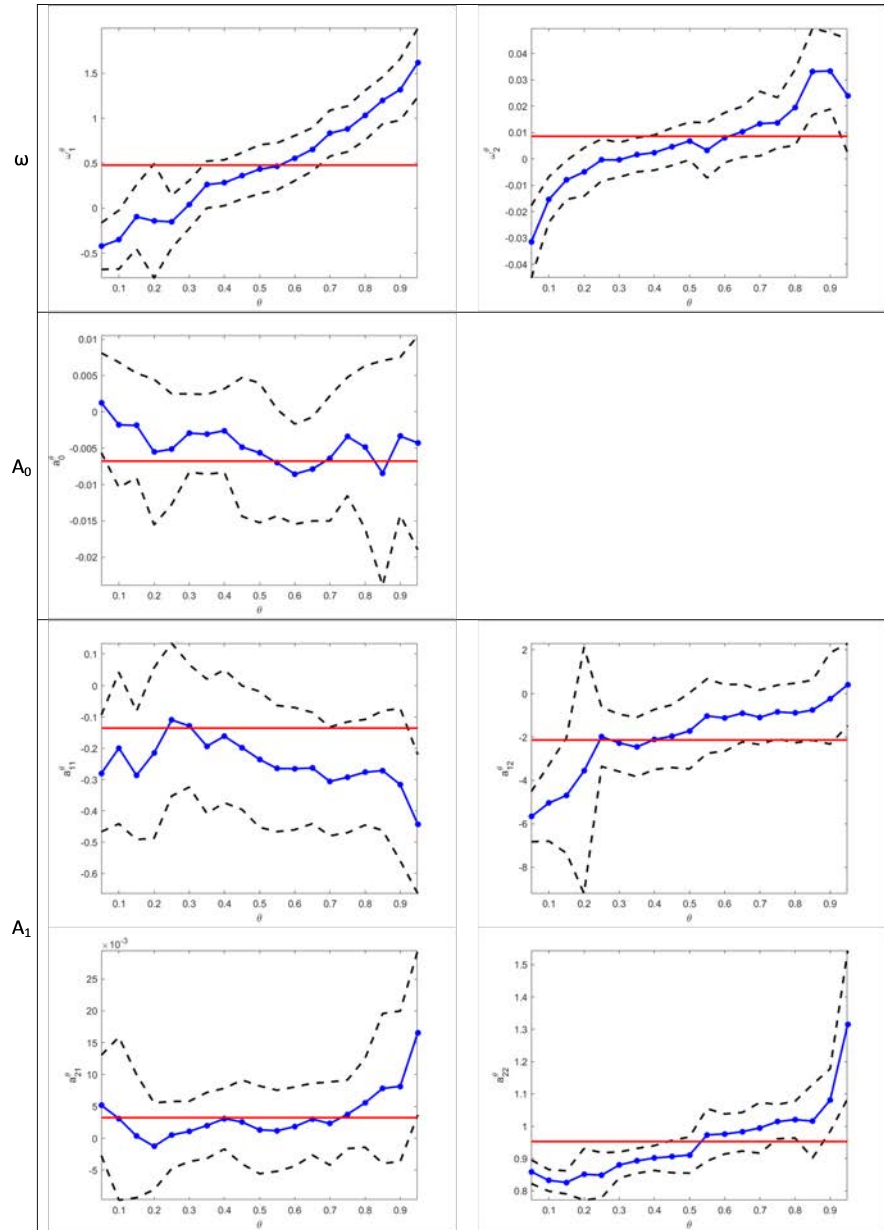
only with a lag. Given the speed at which financial markets react to news, this seems like a reasonable assumption.

The interaction between real and financial variables can be tested by checking whether the off-diagonal coefficients are statistically different from zero. Figure 2 reports the estimated quantile coefficients of (19)-(20), together with 95% confidence intervals and the corresponding OLS estimates. We observe the presence of substantial asymmetries, especially in the a_{12}^θ coefficient, which cannot be detected with standard OLS models. The coefficient estimates of a_{12}^θ are consistent with the findings of Adrian et al. (2019), whereby financial conditions significantly affect the left tail of the distribution of industrial production, but not the right tail.

Figure 3 shows that the impact of financial conditions is not only statistically significant, but also economically relevant. The figure reports the 10% quantile one step ahead forecast of industrial production, together with the 95% confidence intervals. As a comparison, the figure also shows the 10% quantile estimated indirectly from an OLS VAR, obtained as follows. We first estimated the OLS version of model (19)-(20). Second, we computed the 10% quantile of the OLS model residuals and added it to the estimated conditional VAR mean. This procedure would be consistent if model (19)-(20) were correctly specified for the mean and the residuals were i.i.d.

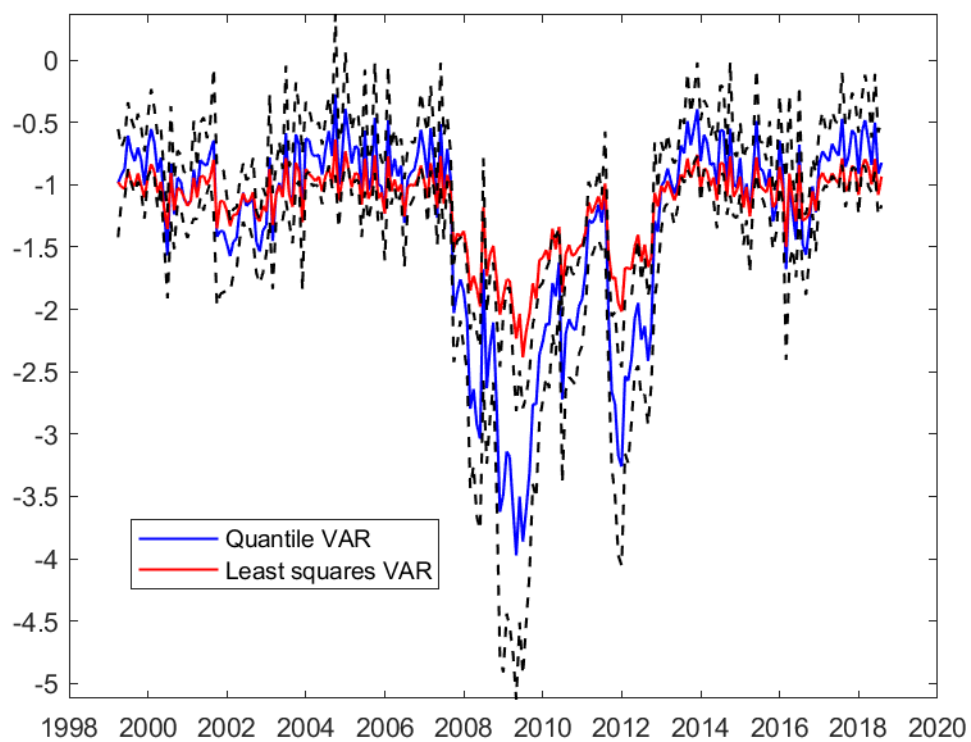
The comparison reveals the strong impact that worsening financial conditions have on the left tail of the forecast distribution. In relation to the OLS estimate, the estimated quantiles are quantitatively and statistically similar

Figure 2: Comparison of QVAR and VAR estimates



Note: Estimated coefficients of model (19)-(20) at different θ quantiles, with 95% confidence intervals. The flat lines represent the corresponding OLS estimates.

Figure 3: Euro area growth at risk



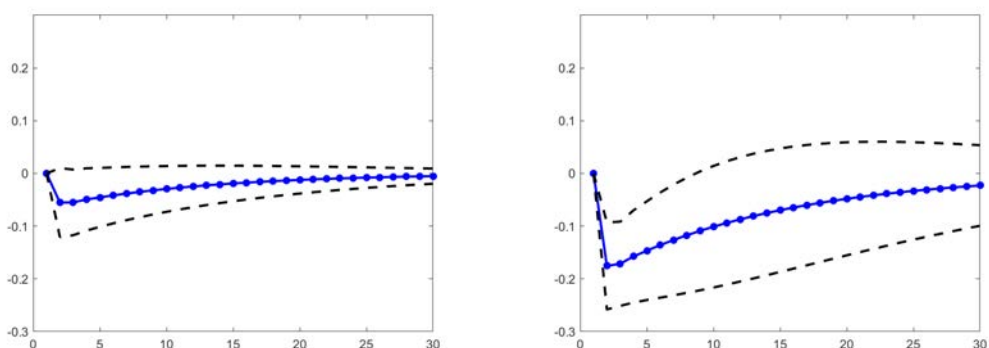
Note: Time series estimates of the 10% quantile of euro area industrial production, together with 95% confidence intervals. As a comparison, it is also reported the 10% quantile estimated by adding to the mean the 10% quantiles of the residuals from a standard OLS VAR. Under correct model specification, the two procedures would give consistent estimates of the 10% quantile. The OLS VAR procedure, however, is not able to capture the asymmetries between financial and real variables.

in tranquil times, but sharply different in crisis times. This highlights how modeling the interactions between real and financial variables with a standard OLS VAR could miss significant dynamics in the left tail, which are relevant from a financial stability perspective.

In figure 4, we compute the quantile impulse response function of industrial production corresponding to (11)-(13), following a one standard deviation shock to CISS structural median residuals and for specific sequences of quantile selection matrices \mathbb{S} . The thought experiment is the following: How different at any point in time the sequence of quantile forecasts would have been if we had observed a more severe realization in the financial conditions of the euro area economy? The left panel is the quantile impulse response function when \mathbb{S} selects the median forecasting path for both endogenous variables. It is the median equivalent to the standard OLS impulse response function for the mean. The QVAR model, however, allows us the flexibility to analyze any part of the forecast distribution, for any period ahead. The right panel of the figure reports the impulse response function when \mathbb{S} selects the 10% quantile of industrial production and the median for CISS. It shows a stronger impact relative to the median.

In figure 5 we report a three dimensional quantile impulse response function. It is a concise way to visualize how each quantile of industrial production is responding to a shock to CISS. It is obtained by stacking next to each other all the panels of Figure 4, when \mathbb{S} selects the median forecasting path for CISS and different values of θ for industrial production. We did not

Figure 4: Quantile impulse response functions for the euro area industrial production



Note: The figure reports how a shock to the financial variable would affect the estimates of future median (left panel) and 10% (right panel) quantiles of euro area industrial production at different time horizons, conditional on a median forecast of the financial variable. 95% confidence intervals are also reported.

report the confidence intervals to avoid cluttering the chart, but they can be readily constructed for each quantile as illustrated in Figure 4. The figure shows on the vertical axis the magnitude of the impulse responses, on the h axis the number of periods for which the response is computed, and on the θ axis the quantile probabilities $\theta \in \{0.05, 0.1, \dots, 0.9, 0.95\}$.

If the OLS VAR model were the correct representation of the dynamic interactions between real and financial variables, all elements of this three dimensional plot would shift in parallel and by the same magnitude across the different quantile probabilities: in an homoskedastic OLS VAR model, shifts in the forecast distribution are entirely driven by changes in the mean forecast. The fact that this does not happen is a further confirmation that OLS VAR may paint a misleading picture when the interest of the analysis is

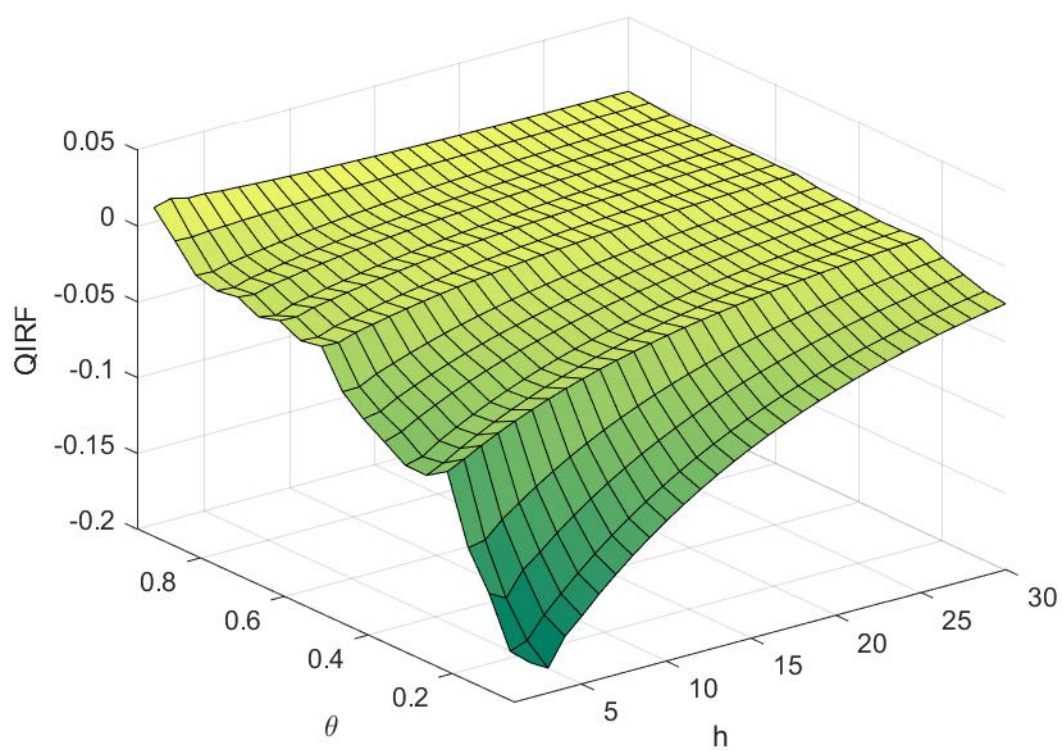
away from the central tendency of the distribution. Consistently with Figure 3, we continue to notice substantial asymmetric impacts in different parts of the distribution. In addition, the chart now reveals that the impact of the shock disappears for all quantiles considered after around 24 months. This analysis highlights one advantage of our framework. It is an internally consistent fully dynamic model of the real and financial variables of the euro area economy, which allows us to study the propagation of shocks across the different parts of the distribution and through time.

4.2 Forecasting growth under stress scenarios

In Figure 6, we report the multi step quantile VAR forecasts of industrial production several months ahead, conditional on many different sequences \mathbb{S} of the quantile selection matrices. The figure on the left reports the forecast as of September 2008 (the month of Lehman's default). The figure on the right is the forecast as of July 2018. Each dotted line corresponds to alternative specifications for the sequence \mathbb{S} in (9)-(10). The various dots at each point in time can be thought as possible realizations from the distribution of the future random variables.

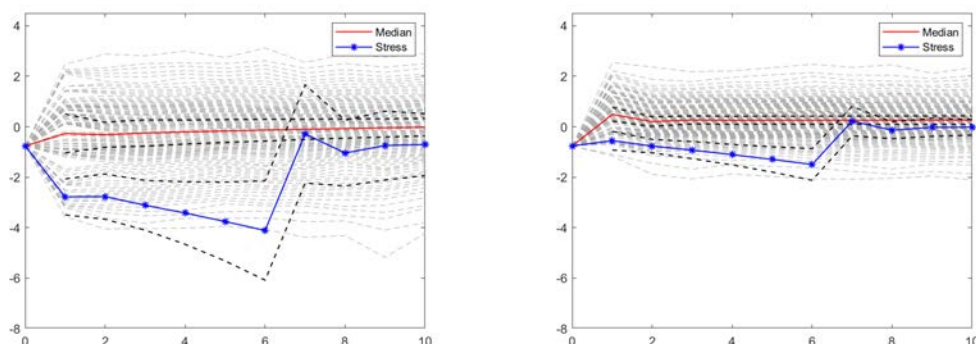
We have highlighted two specific scenarios, both reported with the 95% confidence intervals. The one in blue corresponds to a situation where the sequence of future random variables are set to their median values. This roughly corresponds to the results that one would obtain from a standard OLS VAR analysis. Our framework, however, allows us also to create arbi-

Figure 5: Three dimensional quantile impulse response functions



Note: The figure reports how a shock to the financial variable would affect the estimates of the different quantiles of euro area industrial production at different time horizons, conditional on a median forecast for CISS.

Figure 6: Forecasting and stress testing in the euro area



Note: The figure reports the forecasts of industrial production for the euro area associated with different scenarios. The path highlighted in blue corresponds to a scenario where both the real and financial variables evolve according to their median values. The path highlighted in red corresponds to the stress scenario with a 90% quantile forecast for the financial variable and a 10% quantile forecast for the real variable for six consecutive months, followed by median forecasts afterwards. The panel on the left is the forecast as of August 2008, the panel on the right as of July 2018. 95% confidence intervals are reported around each scenario.

trary stress scenarios and to assess their impact. In the same figure, we have highlighted in red the forecast of the system associated with the following stress testing exercise. We assume that the euro area economy is hit by a series of six consecutive 90% quantile realizations to its financial system and 10% quantile realizations to its real economy. After that, we assume that the system is reverting to normal functioning, by imposing median realizations for all the variables. We notice that the median scenario is very similar at the two points in time considered in this exercise. The stress scenario, however, sees a much more severe contraction in industrial production in August 2008, peaking at about -4%, than in July 2018, where the peak is around -2%.

4.3 Counterfactual scenario analysis of Lehman Brothers' default

One year after the collapse of Lehman Brothers, Queen Elizabeth II famously asked: *Why did nobody notice it?* From the perspective of the methodology of this paper, predicting a crisis and its severity is like predicting that a certain sequence \mathbb{S} of adverse quantile realizations will hit the system. This is impossible. It is possible, however, to use the QVAR methodology to assess the resilience of an economy to alternative stress scenarios.

We estimate the model (19)-(20) using data only up to August 2008, one month before Lehman's default. For given parameter estimates, we use the system to forecast industrial production six months ahead under the following sequences of \mathbb{S} matrices to define alternative scenarios:

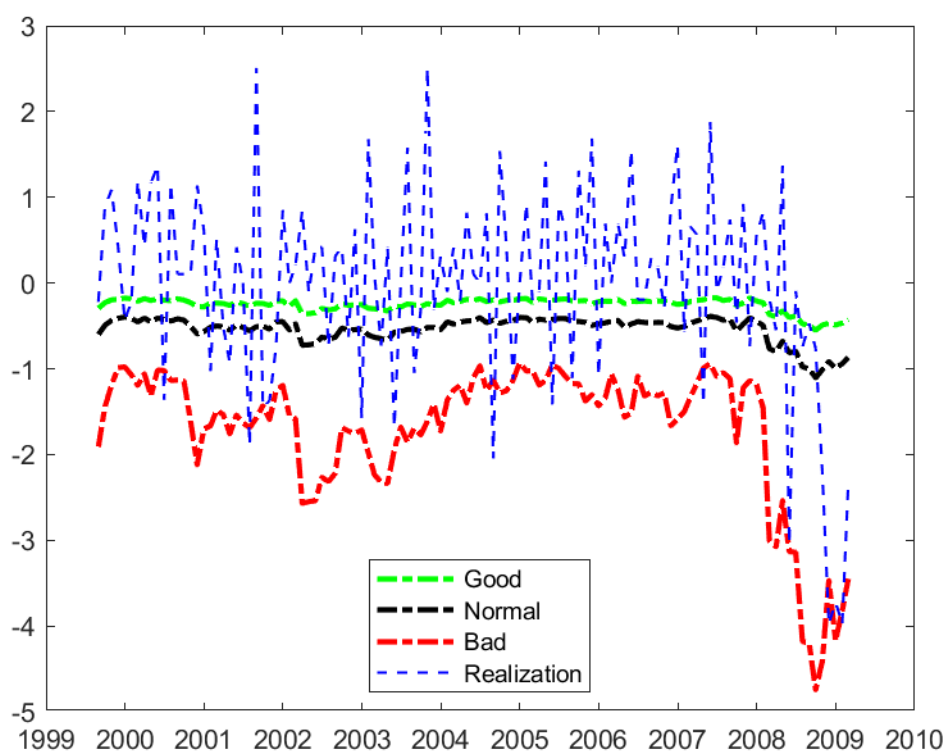
1. **Good financial scenario:** sequence of six 10% quantile realizations for both industrial production and CISS.
2. **Normal financial scenario:** sequence of six 10% quantile realizations for industrial production and median realizations for CISS.
3. **Bad financial scenario:** sequence of six 10% and 90% quantile realizations for industrial production and CISS, respectively.

We apply these scenarios at each month of our sample, and report in figure 7 the six month ahead forecasts for industrial production. It is evident that the good and normal financial scenarios were posing little risks to the

euro area economy, since even after a sequence of six adverse quantile realizations of industrial production, growth at risk was quite contained. It is only under the combination of adverse real and financial quantile realizations that growth at risk is significantly affected. In fact, already in mid 2007, growth at risk under this adverse scenario had reached unprecedented magnitudes for the euro area, from an historical perspective. The large growth at risk under the bad financial scenario reveals the presence of a fat left tail in the distribution of the euro area industrial production, which would go unnoticed by simply estimating the 5% growth at risk using direct estimation techniques.

More generally, such counterfactual exercises are not feasible with the direct forecast approach. By directly quantile regressing industrial production six months ahead against current real and financial conditions, one implicitly imposes that the system evolves according to some average scenario during the intervening six months. While this may be a reasonable assumption if one is interested in modeling the conditional mean of the endogenous variables, it seems like an undesirable constraint to impose when modeling their tail behavior. Notice, however, that if one is interested in such unconditional scenario, this can be recovered from the empirical distribution obtained by simulating the quantile VAR under all alternative quantile scenarios (similarly to all the possible dotted lines of figure 6) and then choosing the desired empirical quantile forecast.

Figure 7: Growth at risk under alternative scenarios as of August 2008



Note: Six month ahead forecast of euro area industrial production under three alternative scenarios. The good, normal and bad scenarios are defined by a sequence of six consecutive benign, normal and adverse quantile realizations. The parameter of the quantile VAR are estimated using only observations up to August 2008.

5 Conclusion

We have developed a quantile VAR model and used it to forecast and stress test the interaction between real and financial variables in the euro area. Unlike OLS VAR, quantile VAR models each quantile of the distribution. This provides the natural modeling environment to design particular stress scenarios and test the impact that they have on the economy. A stress scenario is just a sequence of tail quantile realizations, which can be arbitrarily chosen by the policy maker or calibrated to mimic previous crisis episodes. We find the presence of strong asymmetries in the transmission of financial shocks in the euro area, with negative financial shocks being particularly harmful. By modeling the average interaction between the random variables, OLS VAR models miss most of these detrimental interactions.

Appendix — Proofs

Proof of Theorem 1 (Conditional quantile decomposition of cfd) —

The joint cdf is:

$$\begin{aligned}
F_t(q_{1t}^{\theta_1}, \dots, q_{nt}^{\theta_n}) &= \int_{-\infty}^{q_{1t}^{\theta_1}} \dots \int_{-\infty}^{q_{nt}^{\theta_n}} f_t(y_1, \dots, y_n) dy_1 \dots dy_n \\
&= \int_{-\infty}^{q_{1t}^{\theta_1}} \dots \int_{-\infty}^{q_{nt}^{\theta_n}} f_{1t}(y_1) \dots f_{nt}(y_n) dy_1 \dots dy_n \\
&= \int_{-\infty}^{q_{1t}^{\theta_1}} \dots \int_{-\infty}^{q_{n-1,t}^{\theta_{n-1}}} f_{1t}(y_1) \dots f_{n-1,t}(y_{n-1}) \\
&\quad \left[\int_{-\infty}^{q_{nt}^{\theta_n}} f_{nt}(y_n) dy_n \right] dy_1 \dots dy_{n-1} \\
&= \theta_n \int_{-\infty}^{q_{1t}^{\theta_1}} \dots \int_{-\infty}^{q_{n-1,t}^{\theta_{n-1}}} f_{1t}(y_1) \dots f_{n-1,t}(y_{n-1}) dy_1 \dots dy_{n-1}
\end{aligned}$$

Even though both $q_{nt}^{\theta_n}$ and $f_{nt}(y_n)$ are functions of $(y_1 \dots y_{n-1})$, since we are conditioning on these observations, the probability θ_n associated with $q_{nt}^{\theta_n}$ does not depend on them. It can therefore be pulled out of the integral. The result follows by induction. \square

Proof of Theorem 2 (Law of iterated quantiles) — Start from the innermost expression:

$$Q_{it}^{\vartheta_i}(\epsilon_{1,t+1}^{\vartheta_1} + \dots + \epsilon_{i-1,t+1}^{\vartheta_{i-1}} + \epsilon_{i,t+1}^{\vartheta_i}) = \epsilon_{1,t+1}^{\vartheta_1} + \dots + \epsilon_{i-1,t+1}^{\vartheta_{i-1}}$$

because, by definition $Q_{it}^{\vartheta_i}(\epsilon_{i,t+1}^{\vartheta_i}) = 0$ and the other terms are not random, as they belong to the conditioning set. Repeating this reasoning for each of the remaining terms gives the result. \square

Proof of Theorem 3 (Multi step quantile VAR forecast) — By (6), the forecast of \tilde{Y}_{t+1} , conditional on setting the residuals identified by the matrix $S_{t+1}^{\vartheta^1}$ to zero, is:

$$\begin{aligned}\hat{Y}_{t+1}^{\mathbb{S}} &= S_{t+1}^{\vartheta^1} Y_{t+1} \\ &= S_{t+1}^{\vartheta^1} (\omega + A_0 Y_{t+1} + A_1 Y_t) \\ &= S_{t+1}^{\vartheta^1} \omega + S_{t+1}^{\vartheta^1} A_0 \bar{S} S_{t+1}^{\vartheta^1} Y_{t+1} + S_{t+1}^{\vartheta^1} A_1 Y_t\end{aligned}$$

where we have made use of the equality $\bar{S} S_{t+1}^{\vartheta^1} Y_{t+1} = Y_{t+1}$. Notice that since the vector Y_{t+1} is stacking p times the original vector \tilde{Y}_{t+1} , the operation $S_{t+1}^{\vartheta^1} Y_{t+1} = \tilde{Y}_{t+1}$ implies no loss of information. It is in fact possible to reconstruct Y_{t+1} by stacking again the vector \tilde{Y}_{t+1} with the \bar{S} matrix.

Solving for $S_{t+1}^{\vartheta^1} Y_{t+1}$ and iterating the equation forward, for any given sequence $\{S_{t+h}^{\vartheta^h}\}_{h=1}^H$, we obtain the result. \square

Proof of Corollary 1 (Forecast standard errors) — Consider the mean value expansion $Y_{T+H}(\hat{\beta}) = Y_{T+H}(\beta^*) + \Phi(\bar{\beta})(\hat{\beta} - \beta^*)$. The result follows from the asymptotic properties of $\hat{\beta}$. \square

References

Adrian, T., N. Boyarchenko and D. Giannone (2019), Vulnerable growth, *American Economic Review*, 109 (4), 1262-1289.

Buchinski, M. (1995), Estimating the asymptotic covariance matrix for quantile regression models a Monte Carlo study, *Journal of Econometrics*, 68 (2), 303-338.

Chernozhukov, V., I. Fernandez-Val and A. Galichon (2010), Quantile and probability curves without crossing, *Econometrica*, 78: 1093-1125.

Engle, R.F., D.F. Hendry, J.-F. Richard (1983), Exogeneity, *Econometrica*, 51(2): 277-304.

Engle, R. F. and S. Manganelli (2004), CAViaR: Conditional autoregressive Value at Risk by regression quantiles, *Journal of Business & Economic Statistics*, 22, 367-381.

Gourieroux, C. and J. Jasiak (2008), Dynamic Quantile Models, *Journal of Econometrics*, 147 (1): 198-205.

Hollo, D., M. Kremer and M. Lo Duca (2012), CISS – A composite indicator of systemic stress in the financial system, ECB Working Paper No. 1426.

Hubrich, K. and R. J. Tetlow (2015), Financial stress and economic dynamics: The transmission of crises, *Journal of Monetary Economics*, 70 (C), 100-115.

Kilian, L. and R. J. Vigfusson (2017), The role of oil price shocks in causing U.S. recessions, *Journal of Money, Credit and Banking*, 49 (8), 1747-1776,

Koenker, R. (2005), *Quantile regression*, Econometric Society Monograph, Cambridge University Press.

- Koenker, R. (2017), Quantile regression: 40 years on, *Annual Review of Economics*, 9, 155-176.
- Koenker, R. and G. Bassett (1978), Regression quantiles, *Econometrica*, 46, 33-50.
- Koenker, R. and B. J. Park (1994), An interior point algorithm for nonlinear quantile regression, *Journal of Econometrics*, 71, 265-283.
- Koenker, R. and Z. Hiao (2006), Quantile autoregression, *Journal of the American Statistical Association*, 101 (475), 980-990.
- Lütkepohl, H. (2005), *The New Introduction to Multiple Time Series Analysis*, Springer-Verlag Berlin.
- Mammen, E. (1991), Nonparametric regression under qualitative smoothness assumptions, *The Annals of Statistics*, 19: 741-759.
- Perez-Quiros, G. and A. Timmermann (2000), Firm size and cyclical variation in stock returns, *The Journal of Finance*, LV (3), 1229-1262.
- Schüler, Y. S. (2014), Asymmetric effects of uncertainty over the business cycle: A quantile structural vector autoregressive approach, University of Konstanz, Working Paper Series 2014-02.
- Serfling, R. J. (1980), *Approximation Theorems of Mathematical Statistics*, New York: John Wiley and Sons.

White, H., T.H. Kim and S. Manganelli (2010) Modeling autoregressive conditional skewness and kurtosis with multi-quantile CAViaR, in *Volatility and Time Series Econometrics: Essays in Honor of Robert F. Engle*, edited by T. Bollerslev, J. Russell and M. Watson, Oxford University Press.

White, H., T.H. Kim and S. Manganelli (2015), VAR for VaR: Measuring tail dependence using multivariate regression quantiles, *Journal of Econometrics*, 187, 169-188.

Acknowledgements

We would like to thank for helpful comments and suggestions Tobias Adrian, David Aikman, Vitor Constancio, Manfred Kremer, Bernd Schwaab, seminar participants at the ECB and BIS, as well as conference participants at the workshop on Financial Econometrics and Empirical Modeling of Financial Markets in Greifswald, the Euro Area Business Cycle Network in Madrid, the 4th international workshop on Financial Markets and Nonlinear Dynamics in Paris, the Deutsche Bundesbank spring conference on Systemic Risk and the Macroeconomy, the annual conference of the Society for Financial Econometrics in Shanghai, the Forecasting and Empirical Methods in Macro and Finance NBER SI group in Boston, Forecasting at Central Banks conference in Ottawa. Niklas Grimm provided helpful research assistance. The views expressed are those of the authors and do not necessarily reflect those of the European Central Bank.

Sulkhan Chavleishvili

European Central Bank, Frankfurt am Main, Germany; email: sulkhan.chavleishvili@ecb.europa.eu

Simone Manganelli (corresponding author)

European Central Bank, Frankfurt am Main, Germany; email: simone.manganelli@ecb.europa.eu

© European Central Bank, 2020

Postal address 60640 Frankfurt am Main, Germany

Telephone +49 69 1344 0

Website www.ecb.europa.eu

All rights reserved. Any reproduction, publication and reprint in the form of a different publication, whether printed or produced electronically, in whole or in part, is permitted only with the explicit written authorisation of the ECB or the authors.

This paper can be downloaded without charge from www.ecb.europa.eu, from the [Social Science Research Network electronic library](#) or from [RePEc: Research Papers in Economics](#). Information on all of the papers published in the ECB Working Paper Series can be found on the [ECB's website](#).

PDF

ISBN 978-92-899-3899-0

ISSN 1725-2806

doi:10.2866/589324

QB-AR-19-111-EN-N