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Asset prices, wealth inequality, and welfare: safe assets as a solution

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Abstract

Can rising asset prices reduce wealth inequality? This paper builds a continuous-time heterogeneous-agent general equilibrium in which entrepreneurs hold risky private capital and traditional savers hold safe assets. Safe-asset expansions—via financial innovation, public debt, or a stable equity bubble—operate through a single pass-through: they lower entrepreneurs' undiversified risk exposure, compress risk premia, and raise the interest rate. This slows entrepreneurial wealth accumulation and redistributes wealth toward traditional savers, so inequality falls even as risky asset valuations rise. Savers gain unambiguously. Entrepreneurs' welfare is state-dependent: when their wealth share is low, they prefer a higher risk premium and lose from safe-asset expansions; once sufficiently wealthy, they prefer a higher interest rate that protects a larger wealth base and gain.

JEL: D31, G12, E21, E44.

Keywords: Safe assets; Asset prices; Wealth inequality; Interest rates; Welfare.

Non-Technical Summary

Rising asset prices, falling interest rates, and increasing top-wealth shares are often thought to move together because they favor the rich and hurt tra savers. This paper shows the opposite can happen: when the economy expands the supply of safe assets—via financial innovation, public debt, or certain stable bubbles—asset prices can rise while wealth inequality falls and the welfare of less wealthy savers improves.

The model has two groups: entrepreneurs, who invest in risky private capital and save more, and traditional savers, who hold safer assets. A built-in feedback—entrepreneurs' higher returns and saving—can by itself generate the observed joint trends (higher asset prices, lower safe rates, more top-wealth concentration) without clear welfare gains. In contrast, expanding safe assets raises the safe rate and compresses the risk premium, slowing top-wealth accumulation and redistributing toward ordinary savers.

Welfare effects are asymmetric: savers benefit unambiguously from higher safe rates, while entrepreneurs' welfare is state-dependent—poorer entrepreneurs prefer higher risk premia to grow faster; richer entrepreneurs prefer higher safe rates to protect a larger wealth base. Public debt and risk-sharing innovations therefore act as redistributive tools without direct transfers, reframing how rising asset prices affect inequality and welfare.

1 Introduction

Data from the U.S., Europe and beyond over the past few decades shows an increase in asset prices, a fall in interest rates, and higher top wealth inequality.[1] Conventional wisdom connects these three trends: higher asset prices would favor the rich, while lower interest rates hurt the poor, leading to higher top wealth inequality.[2] This paper questions this view. A pivotal yet underexplored question emerges: Under what conditions can rising asset prices *reduce*, rather than exacerbate, wealth inequality? It shows that expanding the supply of safe assets—through financial innovation, public debt, or stable bubbles—can reduce, rather than exacerbate, wealth inequality, redistributing wealth to less wealthy savers and improving their welfare.

This paper analyzes how various drivers of asset price growth differentially affect wealth inequality and welfare. Central to this framework is the role of safe assets as a novel redistributive mechanism, demonstrating how their expansion can reduce wealth inequality and improve welfare for less wealthy savers.

Existing partial equilibrium models cannot fully capture general equilibrium effects on inequality and welfare, as they often overlook endogenous feedback mechanisms and the role of heterogeneous agents in shaping asset markets. Moreover, existing general equilibrium models fail to distinguish between different drivers of rising asset prices—such as endogenous dynamics versus various types of exogenous shocks—and thus their varying impacts on wealth inequality and welfare.[3]

This paper builds a continuous-time dynamic general equilibrium model with heterogeneous agents, incorporating financial frictions, idiosyncratic investment risks, and unequal capital income. The model distinguishes two agent types: entrepreneurs, who invest in risky private equity, earn

[1] On rising asset prices, see Farhi and Gourio (2018), Greenwald et al. (2019), Van Binsbergen (2020), Gormsen and Lazarus (2025); on the secular fall in real interest rates across advanced economies, see Holston et al. (2017); Rachel and Summers (2019); Del Negro et al. (2017); on rising top-wealth inequality in the U.S., Europe, and globally, see Saez and Zucman (2016); Kuhn et al. (2020b); Chancel et al. (2022).

[2] See Kuhn et al. (2020b); Piketty (2014); Mian et al. (2020)

[3] Benhabib et al. (2019) links top wealth to skewed earnings and heterogeneous returns; Fagereng et al. (2025) studies asset price effects and redistribution in a partial equilibrium setting; Greenwald et al. (2021) shows declining interest rates raise financial wealth inequality but does not distinguish between different drivers; Fernández-Villaverde and Levintal (2024) uses a heterogeneous-agent model to study distributional effects of returns but overlooks endogenous feedback; Gomez and Gouin-Bonenfant (2024) shows equity booms amplify top wealth in general equilibrium.

higher returns, but face skin-in-the-game constraints; and traditional savers, who hold safer assets like public equity and bonds.

This paper uses this framework to make three key contributions. First, it establishes endogenous dynamics—internal feedback loops driven by entrepreneurs' higher saving rates and risk premia—as the primary driver of the observed trends—rising asset prices, falling interest rates, and growing wealth inequality—through a tractable dynamic general equilibrium framework that models asset price formation and wealth distribution. Unlike static or partial equilibrium studies, it captures the full transition path and enables welfare analysis of shocks at steady states.

Second, it introduces safe asset expansion as a redistributive mechanism that reduces wealth inequality while raising asset prices. In the model, safe asset expansion is driven by three types of exogenous shocks: first, financial innovation, which relaxes equity constraints or reduces idiosyncratic risks, directly lowering entrepreneurs' risk exposure; second, stable bubbles on public equity, which create safe value by diversifying idiosyncratic risks; and third, public debt, financed by taxing entrepreneurs' realized returns from risky capital, which increases the total safe asset supply. All three forms of safe asset expansion—financial innovation, stable bubbles, and public debt—reduce wealth inequality and lowers risk premium of the risky asset, providing a fresh perspective on fiscal redistribution.

Third, the model delivers a global welfare characterization over the full transition path.[4] Safe asset expansions work through two margins once mapped into entrepreneurs' undiversified exposure: they directly compress the risk premium and, in general equilibrium, they raise the safe rate by easing precautionary demand. The latter unambiguously improves traditional savers' welfare because safe assets dominate their portfolios. For entrepreneurs, the effect is state-dependent. The model yields a simple wealth-share threshold: when entrepreneurs are relatively poor, they prefer a higher risk premium that accelerates wealth accumulation, so safe asset expansions reduce their welfare; once entrepreneurs are sufficiently rich, they prefer a higher safe rate that protects a larger wealth base, so safe asset expansions increase their welfare. Long-run safe rates are pinned by fundamentals, so

[4]Welfare is defined as lifetime expected utility in this paper.

welfare differences are transitional and distributional rather than driven by permanent changes in growth or impatience.

The model works as follows. Along the transition path, wealth endogenously concentrates among entrepreneurs, who earn higher returns from holding risky private capital and save more due to greater patience. As their wealth share grows, the social discount rate—the wealth-weighted average discount rate—falls, inflating risky asset prices and lowering the interest rate. Thus, endogenous dynamics alone successfully replicate the joint trends of rising asset prices, falling interest rates, and increasing wealth inequality.

Without exogenous shocks, these dynamics yield no net welfare gains or losses. In contrast, exogenous interventions that expand safe asset supply reduce the risk premium on risky capital—the excess return entrepreneurs earn over traditional savers—slowing entrepreneurs' wealth accumulation relative to savers and thus reducing wealth inequality. The mechanism is as follows: Safe asset expansion increases the supply of risk-free assets, easing entrepreneurs' precautionary saving motives, raising the interest rate, and compressing the risk premium entrepreneurs earn. This results in slower wealth accumulation for entrepreneurs, narrowing wealth inequality.

Unlike endogenous dynamics, these interventions produce welfare effects. Traditional savers benefit from higher interest rates on safe assets, which dominate their portfolios, enhancing their welfare. For entrepreneurs, lower returns on risky capital reduce welfare by slowing down wealth growth, but reduced risk exposure lowers precautionary saving motives and increases interest rate. From a policy perspective, public debt is a potent tool: it expands safe assets, reduces inequality, and challenges the notion that rising asset prices always favor the wealthy while harming the less wealthy, achieving fiscal redistribution without direct transfers.

1.1 Literature review

This paper contributes to four literatures: the impact of rising asset prices on the wealth distribution and welfare, the asset-pricing and distributional effect of safe assets, rational bubbles sustained by financial frictions, and macro-finance models with imperfect risk-sharing.

A large empirical and theoretical literature shows that asset markets are central to the evolution of top wealth shares. Kuhn et al. (2020a) and Martínez-Toledano (2020) show that asset prices are significant factors in wealth inequality in the US and Spain. Fagereng et al. (2020a) show that capital gains play an important role in saving behavior. Albuquerque (2022) shows that portfolio changes matter for wealth inequality as well. Cioffi (2021) and Xavier (2021) study wealth inequality by incorporating heterogeneity in risk exposure and asset returns in partial equilibrium models. Gomez et al. (2016) studies the role of aggregate risk in shaping wealth inequality and asset prices. Gomez and Gouin-Bonenfant (2024) studies the long-run effect of low interest rate on wealth inequality. This paper is also related to the small but growing literature studying welfare inequality. Fagereng et al. (2025) and Greenwald et al. (2021) study wealth inequality and welfare inequality mainly in a partial equilibrium setting. Fernández-Villaverde and Levintal (2024) studies the distributional effect of asset returns. Relative to this body of work, our contribution is to show within a full general equilibrium that an endogenous wealth–price feedback—where entrepreneurs earn higher return and save more compared to traditional savers—can by itself generate the joint rise in asset prices and top wealth shares alongside falling safe rates, and to demonstrate that the same increase in asset prices can have opposite effects on wealth inequality and welfare depending on whether it is driven by endogenous forces or by exogenous safe-asset expansions.

This paper brings the safe-asset perspective to the inequality question. Evidence of a persistent global shortage of safe assets and depressed safe real rates is well documented Caballero and Farhi (2017); Caballero et al. (2021), and the surge in top saving has been linked to heightened demand for safe stores of value Mian et al. (2020); international models show that suppliers of safe assets earn convenience yields that shape global rates and welfare Farhi et al. (2018). Laudati (2024) shows that shortage of safe asset is associated with rising inequality and Irie (2024) studies the reallocation effect of better risk-sharing in entrepreneur financing. My contribution is to treat expansions in safe-asset supply—via financial innovation, stable bubbles, or public debt—not merely as pricing forces but as a redistributive mechanism that operates without direct transfers: by raising the safe rate and compressing the risk premium, these interventions slow top-wealth accumulation and improve the welfare of the less wealthy group. This safe-asset channel complements the scarcity literature by making its distributional and welfare consequences explicit within a heterogeneous-agent general equilibrium.

A related strand studies rational bubbles under financial frictions. Classic theories Samuelson (1958); Tirole (1985) have been extended to environments with growth, liquidity and collateral constraints Martin and Ventura (2012); Farhi and Tirole (2012); Martin and Ventura (2018), and recent work shows how bubbles can accommodate high public debt or relax borrowing limits Reis (2021); Brunnermeier et al. (2020); Miao and Wang (2018). I contribute by characterizing a stable bubble on public equity that creates safe-asset-like value by easing an equity (skin-in-the-game) constraint rather than a direct borrowing constraint, thereby expanding safe asset supply. Embedding this bubble in our general-equilibrium environment with heterogeneous agents, we show how it lowers precautionary motives, compresses risk premia, and—unlike most prior analyses—alters the distribution of wealth and welfare; in doing so, we connect to related insights on bubble-created safe value Martin (2016); Miao and Wang (2022) while foregrounding the redistributive effects.

My framework is also related to macro-finance models with financial frictions and idiosyncratic risk Di Tella and Hall (2020); Brunnermeier et al. (2024) and to heterogeneous-agent theories of top-tail inequality driven by heterogeneous returns and saving Benhabib and Bisin (2018); Benhabib et al. (2019); Gabaix et al. (2016). I extend these work by having the specific two-group structure of agents with different constraints and financial market access, and delivering transparent distributional and welfare predictions along the entire transition for both endogenous wealth–price feedback and exogenous safe asset expansions. In this way, I connect the transition path and long-run inequality dynamics in a single, policy-relevant framework centered on the supply of safe assets.

This paper is organized as follows. Section 2 presents motivating facts. Section 3 sets up the model. Section 4 characterizes the fundamental equilibrium. Section 5 introduces the bubble equilibrium. Section 6 studies public debt as a safe asset. Section 7 analyzes safe-asset expansions and their effects on inequality, asset prices, and rates. Section 8 provides welfare analysis. Section 9 concludes.

2 Motivating Facts

Motivated by the well-established fact in the literature that wealth inequality is rising and exploding at the top end in recent decades, as well as a long strand of research showing that asset valuations have also been rising in recent decades, I study how financial assets affect top inequality. This section provides some motivations and rationales for key modelling elements.

A focus on capital A large literature studies the driving forces of rising wealth inequality at the top. While differences in labor income and saving rates are considered important factors driving the exploding trend in many studies, Del Negro et al. (2017) show that labor income differences only can not explain the wealth concentration at the top and Fagereng et al. (2020a) show that saving rates only differ by wealth groups when capital gains are included. Rising asset prices and capital gains in recent decades have become the focus of a growing literature to understand the wealth concentration at the top. Benhabib et al. (2011) show theoretically that capital income risk, rather than labor income, drives the properties of right tail of wealth distribution. Thus, my model features an economy where capital income uncertainty is one of the key elements determining inequality. I show in appendix that the main result and mechanism still go through in an extension with labor.

Financial market structure and frictions It has been shown that there are systematic differences in portfolio compositions and rates of return along the wealth distribution: The super rich entrepreneurs group is characterized by a heavy portfolio share in high-return assets, especially private business equity, while the savers group holds mainly public equity, such as stock market index fund, and safe assets such as deposit (Fagereng et al. (2020a),Kuhn et al. (2020a), Martínez-Toledano (2020),Xavier (2021), and Albuquerque (2022)).[5] To capture such portfolio heterogeneity, I include three classes of assets in the model: private equity, public equity, and risk-free bond. I also assume restricted participation in equity market: savers cannot hold private equity, but can hold public equity *inactively*. Since the access to private equity market is the access to high-return assets, the restricted private

[5] I focus on financial asset in this paper and do not consider housing explicitly.

equity market participation gives rise to return heterogeneity of different groups. The heterogeneous portfolios and returns arise in the model are consistent with the empirical facts discussed earlier. While I interpret the *inactive* participation of savers in public stock market as their holdings of public equity through pension, which account for a non-negligible proportion in the data for some countries, U.S. for example.

Idiosyncratic risk The important role of idiosyncratic risk in explaining the top wealth concentration has been studied both theoretically and empirically (Campbell et al. (2019), Gomez (2023),Benhabib et al. (2019), Atkeson and Irie (2020), and Gocmen et al. (2025)). Kartashova (2014) has documented that private equities on average earn a premium over public equities due to idiosyncratic risks and such return difference varies with economic fundamentals. Di Tella and Hall (2020) show that idiosyncratic risks affect the return of capital and create inefficient recessions. I incorporate idiosyncratic risks associated with private equities as an important ingredient in the model for asset prices and inequality.

Type dependence and size dependence As shown by Gabaix et al. (2016) and Fagereng et al. (2020b), "type dependence" (persistence heterogeneity in returns) and "size dependence" (positive correlation between return and wealth) are important to generate both the high level and the fast rise of top wealth inequality in the past few decades. In the model, agents are born either to be an entrepreneur or a saver throughout their life. This is the "type dependence" needed to generate wealth inequality. And entrepreneurs hold a portfolio with higher return than savers. This is the "size dependence" needed to make entrepreneurs indeed richer than savers in equilibrium.

3 Model

Preferences The model features two groups of agents: entrepreneurs and traditional savers. Both groups exhibit logarithmic utility for tractability, maximizing their expected lifetime utility. They differ in their time discount rates: entrepreneurs have a discount rate of $\rho^e = \rho - \delta^e$, where $\delta^e > 0$

to reflect their greater patience (i.e., lower discount rate), while savers discount at rate ρ.

Technology The economy is an endowment economy with productive capital (modeled as a tree), where per unit of capital produces a units of output. Only entrepreneurs can manage private capital and run firms, which is subject to idiosyncratic (investment) risks. For an individual entrepreneur i, the private capital k_t^i evolves according to the Ito process:

$$\frac{dk_t^i}{k_t^i} = g dt + \tilde{\sigma} d\tilde{Z}_{i,t} \tag{1}$$

where g represents the expected growth rate of capital, and $\tilde{\sigma}$ is the volatility of the idiosyncratic shock $d\tilde{Z}_{i,t}$, specific to entrepreneur i. These idiosyncratic shocks are independent across entrepreneurs and aggregate to zero, i.e., $\int_0^1 d\tilde{Z}_{i,t} = 0$.

Capital can be traded in the market at an endogenous price q_t per unit. The postulated process for the capital price is:

$$\frac{dq_t}{q_t} = \mu_t^q dt$$

where μ_t^q is determined in equilibrium. Notably, q_t bears no idiosyncratic risk, as it is an aggregate variable.

Financial Market Building on the technology, the financial market comprises private equities, public equities, and risk-free bonds, enabling agents to manage risks and allocate wealth.

Entrepreneurs can issue outside equity to the public stock market, but agency frictions due to incentive problems impose a limit: they must retain at least a fraction $\underline{\chi}$ of their firm's value as private equity (the "skin-in-the-game" constraint).[6] The value of outside equity issued by entrepreneur i is $V_t^{oe,i} = (1 - \chi_{it}) q_t k_t^i$, where $1 - \chi_{it}$ is the fraction issued.

Upon pooling outside equities in the public stock market, idiosyncratic risks cancel out, forming a diversified index fund (e.g., akin to the S&P 500) free of idiosyncratic risks. This public equity, or

[6]This equity constraint is widely used in the macro-finance literature, micro-founded in corporate finance, and supported by empirical evidence.

stock market index fund, has a total value V_t^{mf} determined in equilibrium. Traditional savers cannot hold private capital but can hold public equity passively. This can be viewed as holding index funds through pensions.The fraction of public equity held by traditional savers is $1-\kappa$, where κ is a model parameter. Market clearing implies entrepreneurs hold a κ fraction of public equity.

Both groups of agents can trade risk-free bonds B_t, which are in zero net supply.

Figure 1 illustrates the balance sheets of entrepreneurs and savers while figure 2 shows the overall financial market structure.

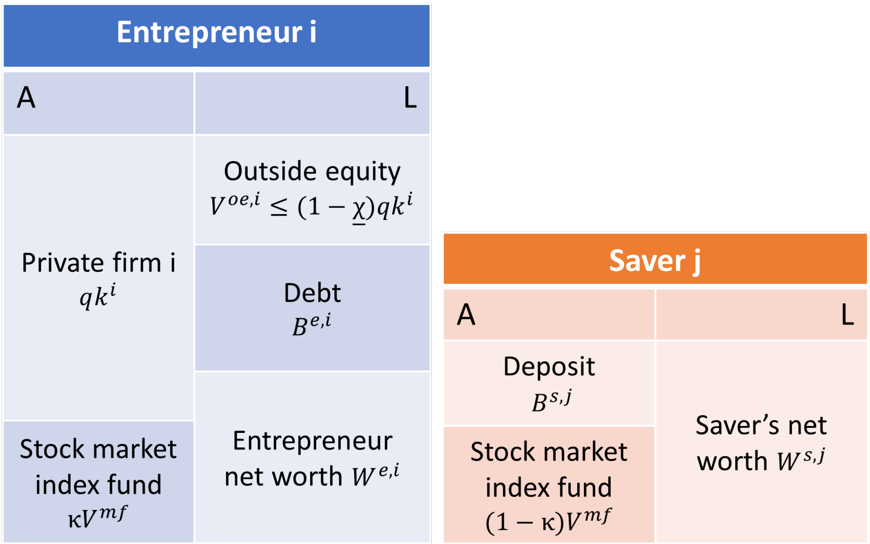

Figure 1: Balance sheets of entrepreneur and saver

Asset returns To analyze agents' investment decisions, I introduce notations for endogenous returns on various assets in the model. The total return on firm i's capital—capital exposed to idiosyncratic risks—is given by:

$$dr_t^{k,i} = \underbrace{\frac{a}{q_t}}_{\text{dividend yield}} dt + \underbrace{\frac{d(q_t k_t^i)}{q_t k_t^i}}_{\text{capital gain}} \tag{2}$$

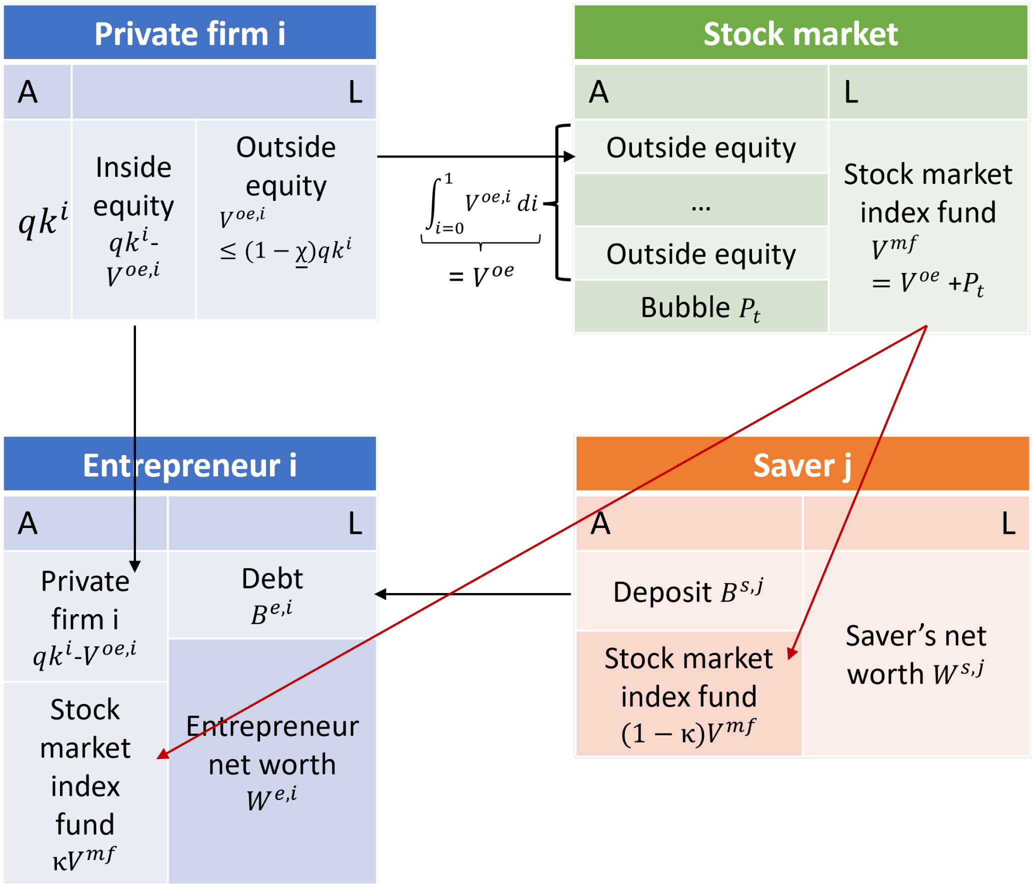

Figure 2: Financial market structure

The return on outside equity issued by entrepreneur i mirrors the risk profile of private (inside) equity but may differ in expected return due to equity constraints:

$$dr_t^{oe,i} = \mathbb{E}[dr_t^{oe,i}] + \tilde{\sigma} d\tilde{Z}_t^i$$

where the expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$ is determined in equilibrium. In the absence of binding equity constraints, $\mathbb{E}[dr_t^{oe,i}] = \mathbb{E}[dr_t^{k,i}]$, aligning with private equity's expected return. However, when constraints bind, outside equity yields a lower expected return: $\mathbb{E}[dr_t^{oe,i}] < \mathbb{E}[dr_t^{k,i}]$.

Public equity, represented by the diversified stock market index fund, has a return dr_t^{mf}, while the risk-free bond offers $dB_t/B_t = r_t^f dt$. Both are uniform across agents, free of idiosyncratic risks.

Wealth and portfolio shares Building on these asset returns, agents allocate their wealth across portfolios to optimize consumption and savings. Let $W_t^{e,i}$ denote the wealth of entrepreneur i and $W_t^{s,j}$ the wealth of traditional saver j.

For entrepreneurs, the portfolio share in private capital—representing the total value of firm i—is $\theta_t^{k,i} = q_t k_t^i / W_t^{e,i}$. The share in outside equity (issued, hence negative) is $\theta_t^{oe,i} = -(1-\chi_{it}) q_t k_t^i / W_t^{e,i}$. Consequently, the effective share in private equity (retained portion, or inside equity) is $\chi_{it} q_t k_t^i / W_t^{e,i}$.

The portfolio share of public equity (the stock market index fund) held by entrepreneur i is denoted as $\theta_t^{mf,i}$, which entrepreneurs optimally choose. In contrast, the portfolio share of public equity held by traditional saver j is $\alpha_t^{mf,j} = \frac{(1-\kappa)V_t^{mf}}{W_t^{s,j}}$, taken as given due to traditional savers' passive holding and not that the parameter κ is identical across all traditional savers.

Demographics To ensure a non-degenerate long-run wealth distribution, I introduce new birth of traditional savers. Let S_t be the set of traditional savers alive at t, with the measure of M_t^s growing at rate $\delta^b > 0$ (newborns only, no deaths):

$$\frac{d \ln M_t^s}{dt} = \delta^b, \quad M_0^s = 1$$

Newborn traditional savers are *endowed* with the current average wealth $\bar{w}_t^s$,

$$\bar{w}_t^s \equiv \frac{W_t^s}{M_t^s},$$

where $W_t^s = \int_{j \in S_t} W_t^{s,j} dj$ is aggregate wealth of traditional savers alive at t. The measure of entrepreneur group is constant and normalized to 1. Their aggregate wealth is $W_t^e = \int_i W_t^{e,i} di$.

Wealth inequality Define entrepreneurs' wealth share

$$\eta_t \equiv \frac{W_t^e}{W_t^e + W_t^s}.$$

I refer to η_t as wealth inequality. An increase in η_t indicates a larger entrepreneur share in total wealth.

Optimization problems The optimization problem for entrepreneur i is as follows:

$$\begin{aligned}
\max_{\{c_t^{e,i},\theta_t^{k,i},\theta_t^{oe,i},\theta_t^{mf}\}_{t=0}^{\infty}} \quad & \mathbb{E}\left[\int_0^\infty e^{-\rho^e t}\log c_t^{e,i}\,dt\right] \\
\text{s.t.} \quad & \frac{dW_t^{e,i}}{W_t^{e,i}} = r_t^f dt + \theta_t^{k,i}(dr_t^{k,i} - r_t^f dt) + \theta_t^{oe,i}(dr_t^{oe,i} - r_t^f dt) + \theta_t^{mf,i}(dr_t^{mf} - r_t^f dt) \\
& \qquad - \frac{c_t^{e,i}}{W_t^{e,i}}dt - (1-\eta_t)(\theta_t^{k,i} + \theta_t^{oe,i} + \theta_t^{mf,i})\delta^b dt \\
& -\theta_t^{oe,i} \le (1-\underline{\chi})\theta_t^{k,i}
\end{aligned} \tag{3}$$

Each entrepreneur optimally selects their consumption plan and portfolio shares in private (inside) equity, outside equity, public equity, and risk-free bonds, taking asset returns as given. The last term, $-(1-\eta_t)(\theta_t^{k,i} + \theta_t^{oe,i} + \theta_t^{mf,i})\delta^b$, is a dilution wedge on the entrepreneur's wealth.[7]

The optimization problem for traditional saver j is:

$$\begin{aligned}
\max_{\{c_t^{s,j}\}_{t=0}^{\infty}} \quad & \int_0^\infty e^{-\rho t}\log c_t^{s,j}\,dt \\
\text{s.t.} \quad & \frac{dW_t^{s,j}}{W_t^{s,j}} = r_t^f dt + \alpha_t^{mf,j}(dr_t^{mf} - r_t^f dt) - \frac{c_t^{s,j}}{W_t^{s,j}} - \alpha_t^{mf,j}(1-\eta_t)\delta^b dt
\end{aligned} \tag{4}$$

Each saver optimally chooses their consumption and saving plan, taking the risk-free rate as given.[8] The last term, $-\alpha_t^{mf,j}(1-\eta_t)\delta^b$, is a dilution wedge on the saver's wealth.[9] The Hamilton-Jacobi-Bellman (HJB) equations for these optimization problems and the associated first-order conditions are detailed in the appendix.

[7]Entry of endowed newborn traditional savers expands aggregate saver wealth and hence saving demand at rate δ^b. This pure transfer is a negative payout funded by existing capital, reducing the instantaneous return on outstanding equities.

[8]Note that savers bear no idiosyncratic risks, as the stock market index fund diversifies them away.

[9]This is a similar dilution wedge on the capital (public equity) held by traditional savers.

Market clearing condition The consumption goods market clears according to:

$$C_t^e + C_t^s = aK_t \tag{5}$$

The left-hand side of equation (5) captures total demand, where $C_t^e = \int_i c_t^{e,i}\, di$ and $C_t^s = \int_j c_t^{s,j}\, dj$ denote the aggregate consumption of entrepreneurs and traditional savers, respectively. The right-hand side reflects total supply, with $K_t = \int_i k_t^i\, di$ as aggregate capital and aK_t as total production at time t.

Definition of bubble The bubble in the model is defined as:

$$P_t = V_t^{mf} - \int_i V_t^{oe,i}\, di \tag{6}$$

Here, V_t^{mf} represents the total market value of public equity, as determined in equilibrium. The second term, $\int_i V_t^{oe,i}\, di$, aggregates the values of all outside equities issued by entrepreneurs, capturing the fundamental value of public equity. Thus, P_t quantifies the endogenous wedge between the market value and fundamental value of public equity, which I term the *bubble* value.

For analytical convenience, I also define $p_t \equiv \frac{P_t}{(1-\chi_t)K_t}$ where $\chi_t K_t = \int_i \chi_{it} k_{it}\, di$. The postulated process for p_t, solved in equilibrium, is:

$$\frac{dp_t}{p_t} = \mu_t^p\, dt.$$

Note that p_t exhibits no idiosyncratic risks, as the bubble value is an aggregate phenomenon.

Throughout the remainder of the paper, superscripts $\{f, b\}$ distinguish variables in the fundamental equilibrium (Section 4) and bubble equilibrium (Section 5), where necessary. Overlines denote steady state values [10].

[10] Here and elsewhere, "steady state" refers to a balanced-growth-path steady state.

4 Fundamental Equilibrium

This subsection defines the fundamental equilibrium, highlights the core links between consumption, asset returns, and the evolution of the wealth distribution, derives asset prices and the risk-free rate, and characterizes the steady state and the transition. Derivations are in the appendix; the bubble equilibrium appears in Section **??**.

Definition 1 (Fundamental Equilibrium)**.** *A fundamental equilibrium is an equilibrium in which the bubble has zero value ($P_t = 0$). Formally, it consists of processes for the capital price q_t, outside equity return dr_t^{oe}, risk-free rate r_t^f, public equity return dr_t^{mf}, and entrepreneurs' wealth share η_t^f, given exogenous parameters $\{\delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g, \delta^b\}$, such that:*

1. *Entrepreneurs solve their optimization problem* (3)*.*
2. *Traditional savers solve their optimization problem* (4)*.*
3. *The consumption-goods market clears* (5)*.*
4. *The bubble value is zero ($P_t = 0$).*

Key equations for the fundamental equilibrium. The fundamental equilibrium is characterized by several key relationships (derivations are in the appendix).

First, by the first-order conditions for consumption, both entrepreneurs and traditional savers consume a constant fraction of their wealth under logarithmic utility:

$$\frac{c_t^{e,i}}{W_t^{e,i}} = \rho - \delta^e \equiv \rho^e, \qquad \frac{c_t^{s,j}}{W_t^{s,j}} = \rho. \tag{7}$$

In words, an entrepreneur consumes a fraction $\rho^e = \rho - \delta^e$ of wealth, while a traditional saver consumes fraction ρ of wealth. Entrepreneurs are assumed more patient than traditional savers ($\delta^e > 0$), so $\rho^e < \rho$ and they consume a smaller share of their wealth, saving more.

Next, because entrepreneurs face a binding equity constraint (they must retain at least fraction $\underline{\chi}$ of the firm as inside equity), private inside equity (the entrepreneur's stake in the firm) commands a higher expected return than outside equity. At the same time, in the absence of a bubble ($P_t = 0$), the expected return on outside equity aligns with the public equity return, which equals the (demographic adjusted) risk-free rate by a no-arbitrage condition (since the public equity portfolio diversifies idiosyncratic risk):

$$\mathbb{E}_t\left[\frac{dr_t^{k,i}}{dt}\right] > \mathbb{E}_t\left[\frac{dr_t^{oe,i}}{dt}\right] = \mathbb{E}_t\left[\frac{dr_t^{mf}}{dt}\right] = r_t^f + (1-\eta_t^f)\delta^b. \tag{8}$$

We next describe the evolution of the key endogenous state variable, the entrepreneurs' wealth share η_t^f, which tracks wealth inequality between entrepreneurs and traditional savers:

$$\frac{d\eta_t^f}{\eta_t^f} = (1-\eta_t^f)\left(\delta^e + \left(\frac{\underline{\chi}\,\tilde{\sigma}}{\eta_t^f}\right)^2 - \delta^b\right)dt. \tag{9}$$

Equation (9) highlights three drivers of wealth inequality: (i) the patience gap δ^e between entrepreneurs and traditional savers; (ii) the risk premium term $\left(\underline{\chi}\tilde{\sigma}/\eta_t^f\right)^2$, which is the excess return entrepreneurs earn on private capital for bearing undiversifiable idiosyncratic risk; and (iii) the birth rate of traditional savers δ^b, which prevents entrepreneurs' dominance in the long-run wealth distribution.

Finally, valuation follows from consumption-goods market clearing. Combining the constant consumption shares in (7) with consumption-goods market clearing (5) pins down the economy's intertemporal pricing at the wealth-weighted discount rate

$$A(\eta_t^f) \equiv \rho - \delta^e\eta_t^f. \tag{10}$$

So the present-value relation from market clearing is

$$A(\eta_t^f)\,q_t = a \qquad \Longrightarrow \qquad q_t = \frac{a}{\rho - \delta^e\eta_t^f}. \tag{11}$$

As entrepreneurs' weight rises, the effective discount rate $\rho - \delta^e\eta_t^f$ falls and valuations rise. This is a purely distributional (composition) channel: no change in productivity a or growth g is required for a valuation boom.

The risk-free rate is then linked to inequality and valuation dynamics. Using either group's Euler equation together with market clearing and the valuation link above yields the identical pricing identity

$$r_t^f \;=\; \rho - \delta^b + g \;-\; \frac{\eta_t^f}{1-\eta_t^f}\,\mu_t^{\eta,f} \;+\; \mu_t^q \;=\; \rho - \delta^b + g \;+\; B(\eta_t^f)\,\mu_t^{\eta,f}, \tag{12}$$

where

$$B(\eta) \;=\; \frac{\delta^e\eta}{\rho-\delta^e\eta} - \frac{\eta}{1-\eta} \;=\; -\frac{(\rho-\delta^e)\,\eta}{(1-\eta)(\rho-\delta^e\eta)} \tag{13}$$

maps the drift of the entrepreneurial wealth share into changes in the risk-free rate. The long-run level $\rho - \delta^b + g$ is the rate implied once the wealth distribution stops drifting.

A pointwise sufficient statistic: entrepreneurs' undiversifiable risk exposure Γ. Define

$$\Gamma_t \;\equiv\; \frac{\underline{\chi}\,\tilde{\sigma}}{\eta_t^f}, \qquad \Delta \;\equiv\; \delta^b - \delta^e \;>\; 0.$$

Then the evolution of η_t^f can be written as

$$\frac{d\eta_t^f}{\eta_t^f} \;=\; (1-\eta_t^f)\Big(\Gamma_t^2 - \Delta\Big)dt. \tag{14}$$

Economically, Γ_t is entrepreneurs' undiversifiable risk exposure per unit of wealth. A higher Γ_t strengthens precautionary saving motive and the entrepreneurs' excess return coming from risk premium; it also pins down whether the entrepreneurial share rises or falls and how fast: the share rises if $\Gamma_t^2 > \Delta$ and falls if $\Gamma_t^2 < \Delta$.

Having derived the key equations, we now characterize the steady state and transition dynamics.

Proposition 1 (Fundamental Equilibrium Steady State ($\delta^e < \delta^b$))**.** *In the steady state of the fundamental equilibrium (no bubble), given parameters* $\{\delta^b, \delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g\}$ *with* $\Delta \equiv \delta^b - \delta^e > 0$, *the capital price, wealth distribution, and bubble value are:*

$$\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\Delta}}, \tag{15}$$

$$\overline{q}^f = \frac{a}{\rho - \delta^e \, \overline{\eta}^f} = \frac{a}{\rho - \delta^e \, \frac{\underline{\chi}\,\tilde{\sigma}}{\sqrt{\Delta}}}, \tag{16}$$

$$\overline{p}^f = 0. \tag{17}$$

To ensure a non-degenerate steady-state wealth distribution in which both entrepreneurs and traditional savers hold positive wealth and the economy discounts the future at a positive rate, require

$$0 < \overline{\eta}^f < \min\left\{1, \frac{\rho}{\delta^e}\right\} \iff \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\Delta}} < \min\left\{1, \frac{\rho}{\delta^e}\right\}.$$

In steady state, the risk-free rate is

$$\overline{r}^f = \rho - \delta^b + g \tag{18}$$

where g is the exogenous growth rate of the capital stock.

Wealth inequality (steady state). The steady state is pinned down by $\mu^{\eta,f} = 0$, equivalently $\Gamma^2 = \Delta$ with $\Gamma = \tilde{\sigma}\underline{\chi}/\overline{\eta}^f$. This balance equates the upward forces—higher saving rate of more patient entrepreneurs and their excess return from risk premium on inside equity—with the downward force from the continual entry of traditional savers at rate δ^b. The fixed point $\overline{\eta}^f = \underline{\chi}\tilde{\sigma}/\sqrt{\Delta}$ is exactly the wealth share at which the upward and downward forces offset each other.

Asset price (steady state). The implied asset price at steady state, $\overline{q}^f = a/(\rho - \delta^e \overline{\eta}^f)$, highlights the distributional channel for valuation. A larger entrepreneurial wealth share lowers the effective discount rate $\rho - \delta^e \eta$, which is a wealth-weighted discount rate; valuations are therefore higher when entrepreneurs hold more wealth.

Risk-free rate (steady state). The risk-free rate at steady state, $\bar{r}^f = \rho - \delta^b + g$, follows from either group's Euler equation. From traditional savers' point of view, once the wealth distribution stops drifting ($\mu_t^{\eta,f} = 0$), the time-varying wealth effect vanishes, anchoring the risk-free rate at $\rho - \delta^b + g$.

Proposition 2 (Transition Dynamics in the Fundamental Equilibrium)**.** *Consider the fundamental equilibrium with parameters $\{\delta^b, \delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g\}$, where entrepreneurs are more patient than traditional savers ($\delta^e > 0$) and that $\rho > \delta^e$, and define $\Delta = \delta^b - \delta^e > 0$. Suppose the initial entrepreneur wealth share satisfies $0 < \eta_0^f < \bar{\eta}^f$, with steady state $\bar{\eta}^f = \underline{\chi}\tilde{\sigma}/\sqrt{\Delta}$. Then:*

1. ***Wealth inequality rises.** With*

$$\mu_t^{\eta,f} \equiv \frac{d\eta_t^f}{\eta_t^f\, dt} = (1 - \eta_t^f)\left[\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2 - \Delta\right] \quad \text{and} \quad \Gamma_t \equiv \frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}, \tag{19}$$

we have $\mu_t^{\eta,f} > 0$ for all $\eta_t^f < \bar{\eta}^f$ and thus $\eta_t^f \uparrow \bar{\eta}^f$ monotonically.

2. ***Capital prices rise.** Valuation satisfies*

$$q_t^f = \frac{a}{\rho - \delta^e \eta_t^f}, \qquad \frac{d \ln q_t^f}{dt} = \frac{\delta^e \eta_t^f}{\rho - \delta^e \eta_t^f}\, \mu_t^{\eta,f}, \tag{20}$$

so $dq_t^f/dt > 0$ along the transition and $q_t^f \uparrow \bar{q}^f$.

3. ***The risk-free rate is persistently depressed (and may fall).** The safe rate is*

$$r_t^f = \rho - \delta^b + g + B(\eta_t^f)\, \mu_t^{\eta,f}, \qquad B(\eta) = \frac{\delta^e \eta}{\rho - \delta^e \eta} - \frac{\eta}{1 - \eta} = -\frac{(\rho - \delta^e)\,\eta}{(1 - \eta)(\rho - \delta^e \eta)} \tag{21}$$

Hence $r_t^f < \rho - \delta^b + g$ whenever $\eta_t^f < \bar{\eta}^f$. Its slope decomposes as

$$\frac{dr_t^f}{dt} = B'(\eta_t^f)\frac{d\eta_t^f}{dt} + B(\eta_t^f)\,\frac{d\mu_t^{\eta,f}}{dt}, \tag{22}$$

so r_t^f may initially fall and eventually converges to $\rho - \delta^b + g$ as $\mu_t^{\eta,f} \to 0$.

Wealth inequality (transition). By (19), a low entrepreneurial share implies high undiversified risk exposure Γ_t and a positive drift $\mu_t^{\eta,f}$, so η_t^f rises monotonically toward its steady state level $\overline{\eta}^f$. The mechanism is the same as in steady state but out of balance: entrepreneurs' higher saving rate and risk premium dominate demographic dilution until their undiversified risk exposure Γ_t^2 falls back to Δ as their wealth accumulates.

Asset price (transition). Using (20), asset prices rise purely through the distributional discounting channel: as η_t^f increases, the wealth-weighted discount rate $\rho - \delta^e \eta_t^f$ falls and q_t^f increases, without any change in cash flows.

Risk-free rate (transition). Equation (21) shows a distribution wedge $B(\eta_t^f)\mu_t^{\eta,f}$ below the long-run anchor $\rho - \delta^b + g$; with $\eta_t^f < \overline{\eta}^f$ this wedge is negative (assuming $\rho > \delta^e$), keeping r_t^f along the transition path lower than its long-run level. The slope decomposition in (22) separates the composition effect $B'(\eta_t^f)\frac{d\eta_t^f}{dt}$ from the weakening inequality engine $B(\eta_t^f)\,\frac{d\mu_t^{\eta,f}}{dt}$: the former can push r_t^f down early on, while the latter closes the wedge as $\mu_t^{\eta,f}$ shrinks as wealth inequality increases to the long-run steady state level.

Interpreting the transition. Together, (19)–(21) imply a unified path: When entrepreneurs wealth share is lower than steady state level, high Γ_t at low η_t^f drives rising inequality, a valuation boom via lower discounting, and a depressed safe rate; as Γ_t falls, these forces weaken and all variables converge to their steady-state benchmarks.

Summary. Taken together, the fundamental equilibrium predicts a clear co-movement: rising wealth inequality, rising valuations, and a falling or persistently low risk-free rate that eventually normalizes as the distribution stabilizes. The bubble equilibrium follows next in Section 5.

5 Bubble Equilibrium

This section examines the bubble equilibrium, in which a stable bubble (a non-fundamental component of asset value) emerges on public equity as a mechanism to expand the supply of safe assets. In the context of this model, the bubble is a rational phenomenon: it has no intrinsic dividend but is valued because agents expect it to persist and provide a store of value. It is “speculative” only in the technical sense that its value is sustained by expectations rather than fundamentals. I first provide intuition for how and why a bubble can form in this model. I then formally define and solve the bubble equilibrium, highlighting the differences from the fundamental equilibrium.

5.1 Intuition for Bubbles

This subsection explains why a bubble can exist in this environment, clarifies the contribution relative to the literature, and frames the bubble as a market-created safe asset.

Why a bubble can exist (intuition). A bubble arises when a binding equity constraint $\chi_{it} \geq \underline{\chi}$ prevents entrepreneurs from fully selling or diversifying idiosyncratic risk, heterogeneous agents face different discount rates, and safe assets are scarce from the perspective of traditional savers. Because entrepreneurs must retain at least $\underline{\chi}$ of their firms, the supply of assets that traditional savers regard as safe is insufficient, creating excess demand for safe stores of value. In such a state, a rational valuation component attached to public equity is sustained as a safe store of value even though it is not backed by dividends. Recall that the economy’s effective discount rate $A(\eta) \equiv \rho - \delta^e \eta$ and write $A_t = A(\eta_t)$. Per unit of capital, total private value satisfies

$$q_t^b + (1 - \underline{\chi})\, p_t = \frac{a}{A_t}, \tag{23}$$

where q_t^b is the fundamental capital price and p_t is the bubble value per unit of outside equity; a positive p_t raises total valuation above fundamental value without relaxing the quantity cap on issuance.

Public equity pricing Public equity is fully diversified against idiosyncratic risk and is therefore priced to the risk-free rate (adjusted for demographic changes) in equilibrium: its return equals $r_t^f + (1 - \eta_t^b)\delta^b$. And the bubble component pays no dividends. If $\omega_t \equiv V_t^{oe}/V_t^{mf}$ is the share of fundamental value in total public equity value, then

$$\mathbb{E}_t[dr_t^{mf}] = \omega_t \mathbb{E}_t[dr_t^{oe}] + (1 - \omega_t)\mathbb{E}_t\Big[\frac{dP_t}{P_t}\Big] = r_t^f \, dt + (1 - \eta_t^b)\delta^b \, dt,, \tag{24}$$

With a bubble (when $p_t > 0$), public equity decomposes into outside equities and the bubble. The bubble component adds additional safe value to public equity. In equilibrium, the return on public equity remains at the risk-free rate (adjusted for demographic changes).

Implication for issuance and selection. Since inside and outside equity now have the same return, entrepreneurs are indifferent at the margin between retaining and issuing another unit of equity. A trembling-hand refinement pins down a unique limit selection in which the constraint binds, $\chi_t = \underline{\chi}$, and $p_t > 0$: the bubble fills the residual demand for safe assets in *value*, not in *quantity*.

Contribution relative to the literature. Unlike models in which bubbles relax borrowing or collateral constraints by expanding financing quantities, the issuance constraint $\chi \geq \underline{\chi}$ remains binding here. The bubble operates on the valuation margin by increasing the value of existing outside equities rather than their quantity, thereby creating safe-asset value endogenously without raising productive capital or issuance flow. This valuation-based safe-asset creation is the channel that lowers entrepreneurs' undiversified risk exposure, slows wealth concentration, compresses risk premia, and moves the risk-free rate transiently. The framework unifies distributional and pricing implications through the single pointwise sufficient statistic Γ_t and makes the bubble's effects directly comparable to the fundamental equilibirum by reducing entrepreneurs' undiversified risk exposure Γ_t.

Bubble as a market-created safe asset. In this setting, the bubble is a non-dividend, riskless valuation component attached to public equity that investors are willing to hold. It supplies the missing safe asset when equity issuance is capped, reallocates wealth from entrepreneurs to traditional savers by reducing undiversified risk exposure Γ_t, and consequently alters inequality dynamics, asset prices and risk-free rate.

5.2 Bubble Equilibrium (Formal Results)

Definition 2 (Bubble Equilibrium)**.** *A bubble equilibrium consists of processes for the capital price q_t, outside equity return dr_t^{oe}, risk-free rate r_t^f, public equity return dr_t^{mf}, and entrepreneurs' wealth share η_t^b, given exogenous parameters $\{\delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g, \kappa\}$, such that:*

1. *Entrepreneurs solve their optimization problem* (3).
2. *Traditional savers solve their optimization problem* (4).
3. *The consumption-goods market clears* (5).
4. *The bubble has positive value: $P_t > 0$ for all t.*
5. *The equilibrium is trembling-hand perfect.*

Key equilibrium relationships: Consumption In the bubble equilibrium, entrepreneurs and traditional savers follow the same consumption policies as in the fundamental equilibrium (log utility implies each consumes a constant fraction of wealth):

$$\frac{c_t^{e,i}}{W_t^{e,i}} = \rho - \delta^e \equiv \rho^e, \qquad \frac{c_t^{s,j}}{W_t^{s,j}} = \rho,$$

identical to Equation (7).

Return on outside equity With the bubble in place, the binding issuance constraint no longer forces a wedge between inside and outside equity *for entrepreneurs*. Therefore,

$$\mathbb{E}_t\left[\frac{dr_t^{k,i}}{dt}\right] = \mathbb{E}_t\left[\frac{dr_t^{oe,i}}{dt}\right] > \mathbb{E}_t\left[\frac{dr_t^{mf}}{dt}\right] = r_t^f + (1-\eta_t^b)\delta^b. \tag{25}$$

The first equality states entrepreneurs expect the same instantaneous return on outside equity as on their inside equity, while public equity remains locally riskless and earns $r_t^f + (1-\eta_t^b)\delta^b$. Relative to the fundamental equilibrium , the bubble allows equal return of inside equity and outside equity, and creates a wedge between the return of outside equity and public equity. Because the bubble absorbs part of traditional savers' demand for safe stores of value, carries a convenience yield, and pays a return *below* the risk. Value-weighted with outside equity, the bubble keeps public equity return locally at $r_t^f + (1-\eta_t^b)\delta^b$ while bidding up total value of public equity V_t^{mf}.

Wealth inequality dynamics The entrepreneurs' wealth share evolves as

$$\frac{d\eta_t^b}{\eta_t^b} = (1-\eta_t^b)\left[(\Gamma_t^b)^2 - \Delta\right]dt, \qquad \Delta \equiv \delta^b - \delta^e > 0, \tag{26}$$

with

$$\Gamma_t^b \equiv \frac{\tilde{\sigma}\,\underline{\chi}}{\eta_t^b}\cdot\frac{q_t^b}{q_t^b + (1-\underline{\chi})p_t} = \frac{\tilde{\sigma}\,\underline{\chi}}{\eta_t^b}\cdot\frac{q_t^b A_t}{a}, \qquad A_t \equiv \rho - \delta^e\eta_t^b,$$

using the accounting identity $(\rho - \delta^e\eta_t^b)\big(q_t^b + (1-\underline{\chi})p_t\big) = a$. The dilution force Δ is unchanged and the patience gap still operates through a lower A_t as η_t^b rises. But the precautionary channel is mitigated because the bubble supplies additional safe assets, reducing the need for entrepreneurs to self-insure by hoarding wealth. Equivalently, the risk premium is lowered as entrepreneurs' undiversified risk exposure is weakened by the value of bubble.

Having these key relationships in place, I proceed to characterize the steady state and transition dynamics of the bubble equilibrium. These results are summarized in the following propositions.

Proposition 3 (Bubble Equilibrium Steady State)**.** *In the steady state of the bubble equilibrium (positive bubble), given parameters* $\{\delta^b, \delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, \kappa\}$ *with* $\Delta \equiv \delta^b - \delta^e > 0$*, the capital price,* *wealth distribution, and bubble value are:*

$$\overline{\eta}^{\,b} = \frac{-\beta \; - \; \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \tag{27}$$

$$\overline{q}^{\,b} = \frac{a\,\overline{\eta}^{\,b}\,\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}\,(\rho - \delta^e\,\overline{\eta}^{\,b})} \tag{28}$$

$$\overline{p}^{\,b} = \frac{a}{\rho - \delta^b\,\overline{\eta}^{\,b}} \; - \; \overline{q}^{\,b} \tag{29}$$

where

$$\alpha \equiv \frac{\delta^b\,\sqrt{\Delta}}{\tilde{\sigma}}, \qquad \beta \equiv -\delta^b \; - \; \rho\,\frac{\sqrt{\Delta}}{\tilde{\sigma}} \; + \; (1 - \underline{\chi})\,\delta^e, \qquad \gamma \equiv \underline{\chi}\,\rho.$$

And the decomposition of fundamental price of capital and bubble value is given by

$$\frac{\overline{q}^{\,b}}{\overline{q}^{\,b} + (1 - \underline{\chi})\,\overline{p}^{\,b}} = \frac{\overline{\eta}^{\,b}\,\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}} \; \in \; (0, 1), \tag{30}$$

To ensure a non-degenerate interior positive-bubble steady state with finite prices, require

$$0 \; < \; \overline{\eta}^{\,b} \; < \; \min\left\{1, \frac{\rho}{\delta^e}\right\}, \qquad 0 \; < \; \frac{\overline{\eta}^{\,b}\,\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}} \; < \; 1, \qquad \Psi \; \geq \; 0,$$

which imply $\overline{q}^{\,b} > 0$ *and* $\overline{p}^{\,b} > 0$*. In steady state, the risk-free rate is*

$$\overline{r}^{\,f} \;=\; \rho - \delta^b + g. \tag{31}$$

The parameter κ *does not affect the interior steady state.*

Wealth inequality (steady state). The steady state is again pinned down by the single condition on entrepreneurs' undiversified risk exposure:

$$\mu^{\eta,b} = 0 \quad \Longleftrightarrow \quad (\Gamma^b)^2 = \Delta, \qquad \Delta \equiv \delta^b - \delta^e > 0.$$

Recall that Γ summarizes how much undiversified risk the entrepreneurial sector bears at the aggregate level, given the equity issuance cap $\underline{\chi}$ and the wealth distribution. With a bubble,

$$\Gamma^b \;=\; \frac{\tilde{\sigma}\,\underline{\chi}}{\eta^b}\cdot\frac{q^b}{q^b+(1-\underline{\chi})p} \;=\; \Gamma^f\cdot\frac{q^b}{q^b+(1-\underline{\chi})p}, \qquad \Gamma^f \equiv \frac{\tilde{\sigma}\,\underline{\chi}}{\eta}.$$

The bubble component $p > 0$ enlarges public equity valuation relative to its fundamental value and thereby reduces exposure through the factor $\frac{q^b}{q^b+(1-\underline{\chi})p} \in (0,1)$. Intuitively, part of the economy's risk-bearing is absorbed by valuation (a larger safe value), so entrepreneurs need not carry as much risk exposure.

Since $(\Gamma^b)^2 = \Delta$ must hold in steady state and $\Gamma^b < \Gamma^f$ at any given η, the same threshold $\sqrt{\Delta}$ is reached at a *lower* entrepreneurial wealth share:

$$\overline{\eta}^{\,b} \;<\; \overline{\eta}^{\,f}.$$

The bubble moderates long-run wealth concentration.

Asset price (steady state). In the bubble equilibrium steady state, asset valuations allocate aggregate saving between risky capital and a safe store: the fundamental risky capital price $\overline{q}^{\,b}$ rises with entrepreneurial risk-bearing capacity—higher $\overline{\eta}^{\,b}$ or lower undiversified risk $\tilde{\sigma}\,\underline{\chi}$—while the bubble $\overline{p}^{\,b}$ absorbs the residual demand for safety, $\overline{p}^{\,b} = a/(\rho - \delta^b\,\overline{\eta}^{\,b}) - \overline{q}^{\,b}$. The composition identity

$$\frac{\overline{q}^{\,b}}{\overline{q}^{\,b}+(1-\underline{\chi})\,\overline{p}^{\,b}} = \frac{\overline{\eta}^{\,b}\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}} \in (0,1)$$

makes the mechanism transparent: the fundamental share equals a simple index of risk-bearing capacity relative to undiversified risk intensity. When capacity is scarce (low $\overline{\eta}^{\,b}$ or high $\tilde{\sigma}\,\underline{\chi}$), the bubble must be larger; when capacity is ample, fundamentals carry more value.

Risk-free rate (steady state). Long-run risk-free rate in bubble equilibrium is the same as in fundamental equilibrium. Once $\mu^{\eta,b} = 0$, the wealth effect vanishes and the risk-free rate is pinned by traditional savers' discounting and demographics as they are the marginal investors of safe assets in this economy:

$$\overline{r}^f \;=\; \rho - \delta^b + g.$$

Proposition 4 (Transition Dynamics in the Bubble Equilibrium)**.** *Consider the bubble equilibrium with parameters* $\{\delta^b, \delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g\}$ *and let* $\Delta \equiv \delta^b - \delta^e > 0$*. Suppose* $0 < \eta_0^b < \overline{\eta}^b$*, where* $\overline{\eta}^b$ *is the unique interior fixed point. Define*

$$A_t \equiv \rho - \delta^e \eta_t^b, \qquad V_t \equiv q_t^b + (1-\underline{\chi})\, p_t = \frac{a}{A_t}, \qquad \theta_t \equiv \frac{q_t^b}{V_t} \in (0,1],$$

and the sufficient statistic (entrepreneurial undiversified risk exposure)

$$\Gamma_t^b \equiv \frac{\tilde{\sigma}\,\underline{\chi}}{\eta_t^b}\theta_t \;=\; \Gamma_t^f \cdot \theta_t, \qquad \Gamma_t^f \equiv \frac{\tilde{\sigma}\,\underline{\chi}}{\eta_t^b}.$$

Then:

1. ***Wealth inequality rises, but more slowly than under fundamentals.***

$$\mu_t^{\eta,b} \equiv \frac{d\eta_t^b}{\eta_t^b dt} = (1-\eta_t^b)\left[(\Gamma_t^b)^2 - \Delta\right]. \tag{32}$$

If $\eta_t^b < \overline{\eta}^b$*, then* $(\Gamma_t^b)^2 > \Delta$ *and* $\eta_t^b \uparrow \overline{\eta}^b$ *monotonically. For any matched* η*,* $\theta_t \in (0,1]$ *implies* $(\Gamma_t^b)^2 < (\Gamma_t^f)^2$ *and hence* $\mu_t^{\eta,b} < \mu_t^{\eta,f}$*, so wealth concentrates more slowly than in the fundamental equilibrium. At the steady state,* $(\Gamma^b)^2 = \Delta$ *with*

$$\theta_\infty \;=\; \frac{\overline{q}^b}{\overline{q}^b + (1-\underline{\chi})\,\overline{p}^b} \;=\; \frac{\overline{\eta}^b\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}} \in (0,1), \qquad \Rightarrow \qquad \overline{\eta}^b < \overline{\eta}^f \equiv \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\Delta}}.$$

2. ***Asset prices: envelope versus composition.*** *Total public valuation is pinned by distributional discounting,*

$$V_t = \frac{a}{A_t} \quad \textit{with} \quad A_t = \rho - \delta^e \eta_t^b, \tag{33}$$

while the split is purely compositional,

$$q_t^b = \theta_t V_t, \qquad p_t = \frac{V_t}{1 - \underline{\chi}} (1 - \theta_t).$$

Growth rates separate the rising envelope V_t from composition:

$$\frac{d \ln q_t^b}{dt} = \frac{\delta^e \eta_t^b}{A_t} \mu_t^{\eta,b} + \frac{d \ln \theta_t}{dt}, \qquad \frac{d \ln p_t}{dt} = \frac{\delta^e \eta_t^b}{A_t} \mu_t^{\eta,b} - \frac{\dot{\theta}_t}{1 - \theta_t}. \tag{34}$$

Thus V_t rises strictly along the path (as A_t falls with η_t^b), while q_t^b and p_t adjust through θ_t to price risk.

3. ***Risk-free rate: a milder distribution wedge.***

$$r_t^{f,b} = \rho - \delta^b + g + B(\eta_t^b)\, \mu_t^{\eta,b}, \qquad B(\eta) = -\frac{(\rho - \delta^e)\, \eta}{(1 - \eta)(\rho - \delta^e \eta)} < 0. \tag{35}$$

Hence $r_t^{f,b} < \rho - \delta^b + g$ whenever $\mu_t^{\eta,b} > 0$, and $r_t^{f,b} \to \rho - \delta^b + g$ as $\mu_t^{\eta,b} \to 0$. For any matched η, $\mu_t^{\eta,b} < \mu_t^{\eta,f}$ implies $r_t^{f,b} > r_t^{f,f}$. The slope decomposition mirrors the fundamental case:

$$\frac{dr_t^{f,b}}{dt} = B'(\eta_t^b) \frac{d\eta_t^b}{dt} + B(\eta_t^b) \frac{d\mu_t^{\eta,b}}{dt}. \tag{36}$$

The parameter κ does not affect these interior laws or the steady-state limits.

Wealth inequality (transition, bubble vs. fundamental). The bubble operates only through composition, via $\theta_t < 1$, which scales down entrepreneurial exposure to undiversified risk: $\Gamma_t^b = \Gamma_t^f \theta_t$. By (32), this reduces the drift of η_t^b at any given wealth distribution, slowing the concentration of wealth relative to the fundamental path. In the long run, the fixed point condition $(\Gamma^b)^2 = \Delta$ with $\theta_\infty \in (0, 1)$ implies a strictly lower steady-state share, $\overline{\eta}^b < \overline{\eta}^f$.

Asset prices (transition: q and p). As in fundamentals, total public value $V_t = a/A_t$ rises purely because the wealth-weighted discount rate $A_t = \rho - \delta^e \eta_t^b$ falls with η_t^b. The difference is compositional: $q_t^b = \theta_t V_t$ and $p_t = (V_t/(1-\underline{\chi}))(1-\theta_t)$. Relative to the fundamental case ($\theta_t \equiv 1$, $p \equiv 0$), a positive bubble tilts valuation toward the safe component. From (34), q_t^b inherits the envelope growth of V_t but is modulated by $\dot{\theta}_t$; likewise p_t grows with the envelope and with any decline in θ_t. Intuitively, the bubble supplies a safe store that absorbs part of aggregate saving, allowing a given V_t to be supported with less risky capitalization q_t^b.

Risk-free rate (transition). Equation (35) shows the same long-run anchor as in fundamentals, $\rho - \delta^b + g$, but with a smaller (less negative) distribution wedge because $\mu_t^{\eta,b} < \mu_t^{\eta,f}$ at matched η. Thus the bubble equilibrium features a safe rate that is less depressed—often higher than in the fundamental path for the same wealth distribution—and it converges to the same limit as the inequality engine $\mu_t^{\eta,b}$ weakens.

Core mechanism. A single statistic governs all dynamics: $\Gamma_t = \frac{\tilde{\sigma}\underline{\chi}}{\eta_t} \times \theta_t$(fundamental share). The bubble lowers the fundamental share θ_t, thereby de-risking entrepreneurs, slowing the rise of wealthinequality, reallocating valuation from q to p, and keeping r^f closer to its anchor throughout the transition.

6 Public Debt as a Safe Asset

Public debt is a government-supplied safe asset. We study how its supply reshapes private portfolios, asset prices, the dynamics of wealth inequality, and the risk-free rate, and how these effects differ between (i) a fundamental equilibrium with no bubble and (ii) a bubble equilibrium with a strictly positive equity bubble. We first define the environment with fiscal policy and bonds, then analyze the fundamental and bubble regimes in turn, highlighting the novel aspects introduced by public debt.

6.1 Environment

Policy instruments and public debt. The government taxes total capital returns at rate $\tau \in [0, 1)$, spends sK_t per unit of capital, and issues risk-free bonds B_t held by private agents. Let $b_t \equiv B_t/K_t$ denote public debt per unit of capital. Bonds pay the endogenous safe rate r_t^f.

Capital returns and taxation. For entrepreneur i, the pre-tax instantaneous (total) return on capital per unit of K_t is

$$dr_t^{k,i} = \left(\frac{a}{q_t} + g + \mu_t^q\right)dt \;+\; \tilde{\sigma}\, d\tilde{Z}_t^i, \tag{37}$$

where a is the cash flow (per unit capital), g the growth of K_t, μ_t^q the expected capital gain on q_t, and $\tilde{Z}_t^i$ an idiosyncratic Brownian motion. The tax applies to the entire return (including capital gains), so the after-tax return scales both drift and diffusion:

$$dr_t^{k,\tau,i} = (1-\tau)\, dr_t^{k,i} = (1-\tau)\left(\frac{a}{q_t} + g + \mu_t^q\right)dt + (1-\tau)\tilde{\sigma}\, d\tilde{Z}_t^i. \tag{38}$$

Government flow budget (in expectation). Tax revenue equals $\tau\left(\frac{a}{q_t} + g + \mu_t^q\right) q_t K_t\, dt$. With spending $sK_t\, dt$ and interest $r_t^f B_t\, dt$,

$$dB_t = r_t^f\, B_t\, dt \;+\; sK_t\, dt \;-\; \tau\left(\frac{a}{q_t} + g + \mu_t^q\right) q_t K_t\, dt. \tag{39}$$

Per unit of capital, using $b_t = B_t/K_t$ and the fact that $dK_t/K_t = g\, dt$,

$$db_t = \left[(r_t^f - g)\, b_t \;+\; s \;-\; \tau\big(a + (g + \mu_t^q) q_t\big)\right] dt. \tag{40}$$

Common notations. Let

$$A(\eta) \equiv \rho - \delta^e \eta, \qquad V(\eta_t) \equiv \frac{a-s}{A(\eta_t)}, \qquad B(\eta) \equiv \frac{\delta^e \eta}{\rho - \delta^e \eta} - \frac{\eta}{1-\eta} = -\frac{(\rho - \delta^e)\,\eta}{(1-\eta)(\rho - \delta^e \eta)}. \tag{41}$$

Here $A_t \equiv A(\eta_t)$ is the wealth-weighted discount rate and $V(\eta_t)$ the total private valuation envelope: government spending s subtracts from the private flow, so $V = (a - s)/A$[11]. With a proportional tax on total returns, taxes operate through risk exposure rather than V. Define $\Delta \equiv \delta^b - \delta^e > 0$ and

$$\theta_t \equiv \frac{q_t}{V(\eta_t)} \in [0, 1], \quad \Gamma_t = \frac{\underline{\chi}(1-\tau)\tilde{\sigma}}{\eta_t}\,\theta_t, \quad \text{and} \quad d\eta_t = \eta_t\, \mu_t^\eta\, dt, \;\; \mu_t^\eta = (1-\eta_t)\big[\Gamma_t^2 - \Delta\big]. \quad (42)$$

The safe rate always admits the distribution-wedge decomposition

$$r_t^f = \rho - \delta^b + g \;+\; B(\eta_t)\, \mu_t^\eta. \quad (43)$$

For later use, $dV_t/V_t = \mu_t^V\, dt$ with

$$\mu_t^V = \frac{\delta^e \eta_t}{A(\eta_t)}\, \mu_t^\eta, \quad (44)$$

since $V = (a - s)/A$ and $dA_t = -\delta^e\, d\eta_t$.

6.2 Fundamental equilibrium with public debt

With $p_t = 0$, public debt b is the unique additional safe asset in the economy. At a given wealth distribution $\eta \in (0, 1)$, fundamentals pin the total value

$$V(\eta) = \frac{a - s}{\rho - \delta^e \eta} \equiv \frac{a - s}{A(\eta)}.$$

Public debt only changes the *composition* of claims between risky equity q^f and safe bonds b; it does not alter $V(\eta)$ on impact.

Proposition 5 (Public debt in the fundamental equilibrium)**.** *Fix $\Delta > 0$ and $\eta \in (0, 1)$. Let $V \equiv V(\eta)$ and $\theta^f \equiv q^f/V$.*

[11] $V(\eta) = a/A(\eta)$ is the same as before in the absence of government.

1. ***Composition (one-for-one).***

$$q^f + b = V, \qquad \theta^f = 1 - \frac{b}{V}. \tag{45}$$

2. ***Risk sharing and inequality.*** *Exposure and inequality drift are*

$$\Gamma^f = \frac{\underline{\chi}(1-\tau)\tilde{\sigma}}{\eta}\,\theta^f, \qquad \mu^\eta = (1-\eta)\big[(\Gamma^f)^2 - \Delta\big], \tag{46}$$

hence at fixed η, higher b lowers Γ^f and μ^η.

3. ***Risk-free rate (local).*** *If $\rho > \delta^e$ so $B(\eta) < 0$,*

$$\left.\frac{\partial r^f}{\partial b}\right|_\eta = B(\eta)\left.\frac{\partial \mu^\eta}{\partial b}\right|_\eta > 0. \tag{47}$$

4. ***Transition with fixed b (amplification).*** *Since $q^f = V - b$,*

$$\frac{dq_t^f}{q_t^f} = \frac{V(\eta_t)}{V(\eta_t) - b}\,\frac{dV_t}{V_t}, \qquad r_t^f = \rho - \delta^b + g + B(\eta_t)\mu_t^\eta. \tag{48}$$

Thus q^f is more sensitive to discount-rate movements by the factor $V/(V-b) > 1$, while the safe-rate wedge is smaller at matched η_t.

5. ***Steady state (closed form).*** *Let*

$$c \equiv \frac{\underline{\chi}(1-\tau)\tilde{\sigma}}{\sqrt{\Delta}} > 0. \tag{49}$$

An interior fixed point $\bar{\eta}^f(b) \in (0,1)$ solving $(\Gamma^f)^2 = \Delta$ exists uniquely for

$$0 \le b < \min\left\{\frac{a-s}{\rho},\ \frac{a-s}{c\,\delta^e}\right\}, \tag{50}$$

and equals

$$\bar{\eta}^f(b) = \frac{c\left(1 - \frac{\rho b}{a-s}\right)}{1 - \frac{c\,\delta^e}{a-s}\,b} \; < \; c \; = \; \bar{\eta}^f(0). \tag{51}$$

At $\bar{\eta}^f(b)$*,*

$$\bar{q}^f = \bar{V} - b, \quad \bar{V} = \frac{a-s}{\rho - \delta^e \bar{\eta}^f(b)}, \quad \bar{r}^f = \rho - \delta^b + g, \quad \bar{\mu}^\eta = 0. \tag{52}$$

What moves on impact (at fixed η). Debt is a pure composition tool: a surprise issuance $db > 0$ swaps risky equity $dq^f = -db$ for safe bonds without altering total value $V(\eta)$. This reduces the risky share θ^f and, with it, entrepreneurs' undiversified risk exposure Γ^f. Two immediate pricing implications follow. First, the risk premium compresses one-for-one with exposure: $d\mu^{k,\text{ex}} = \tilde{\sigma}\, d\Gamma^f < 0$. Second, because precautionary saving motive is weaker when Γ^f falls, the safe rate increases on impact when $\rho > \delta^e$ by (47). None of these changes requires or induces an impact change in $V(\eta)$; they are purely about who holds risk and how much compensation risk-bearing commands.

Risk sharing, inequality, and policy levers. Lower θ^f reassigns idiosyncratic risk from entrepreneurs to the broad saver base, slowing the inequality engine $\mu^\eta = (1-\eta)[(\Gamma^f)^2 - \Delta]$. Debt b and capital-income taxation τ operate through distinct yet complementary channels: b reduces the risky share θ^f (a composition effect), while τ scales down the exposure per unit risky share (a direct effect). Both reduce Γ^f and therefore inequality drift, with parallel positive effects on the safe rate when $\rho > \delta^e$; see (46)–(47).

Transition dynamics and the amplification factor. With b fixed, $q^f = V - b$ mechanically inherits the revaluation of fundamentals but with *higher sensitivity*:

$$\frac{dq^f}{q^f} = \frac{V}{V-b}\frac{dV}{V}, \qquad \frac{V}{V-b} > 1.$$

Intuitively, because part of aggregate value has been carved out into riskless bonds, the remaining risky piece $q^f = V - b$ is a thinner slice that is more "duration-levered" to discount-rate movements driven by μ_t^η. So any revaluation of V caused by discount-rate changes shows up as a larger percentage move in q^f. At matched η_t, the precautionary wedge $B(\eta_t)\mu_t^\eta$ is smaller under higher b (since μ_t^η is lower), keeping r_t^f closer to its anchor $\rho - \delta^b + g$.

Steady state mapping, admissible range, and monotonicity. The fixed-point condition $(\Gamma^f)^2 = \Delta$ maps $(b, \tau, \underline{\chi}, \tilde{\sigma})$ into a unique $\bar{\eta}^f(b)$ given by (51). The bounds in (50) have clear meanings. The first, $b < (a - s)/\rho$, ensures $q^f = V - b \geq 0$ for all $\eta \in (0, 1)$. The second, $b < (a - s)/(c\,\delta^e)$, guarantees that the fixed-point mapping remains interior ($\theta^f > 0$) at the solution. Within this admissible region, $\bar{\eta}^f(b)$ is decreasing in b under the empirically relevant ordering $\rho > c\,\delta^e$ (which is implied by $\rho > \delta^e$ and $c < 1$): more public debt yields safer private portfolios and therefore requires *less* entrepreneurial wealth concentration to satisfy the pricing condition $\Gamma^2 = \Delta$. Regardless of b, the long-run safe rate pins down at its anchor, $\bar{r}^f = \rho - \delta^b + g$, because $\bar{\mu}^\eta = 0$ eliminates the precautionary wedge.

Comparative statics beyond debt. Equation (51) also clarifies how other primitives shift the steady state. A higher tax rate τ (holding b fixed) lowers c and thus reduces $\bar{\eta}^f$; looser financing frictions (lower $\underline{\chi}$) or safer projects (lower $\tilde{\sigma}$) likewise reduce c and make the distribution more egalitarian. Conversely, tighter equity constraint (higher $\underline{\chi}$) or riskier projects (higher $\tilde{\sigma}$) raise c and require a larger entrepreneurial wealth share in steady state. In all cases, the mechanism operates through the same sufficient statistic Γ^f.

Boundary and corner cases. As b approaches its admissible upper bound in (50), the risky slice is crowded out:

$$\theta^f = 1 - \frac{b}{V} \downarrow 0, \qquad q^f = V - b \downarrow 0.$$

At this corner, entrepreneurs bear essentially no idiosyncratic risk ($\Gamma^f = 0$), so the inequality drift turns negative,

$$\mu^\eta = (1-\eta)\left[(\Gamma^f)^2 - \Delta\right] = -(1-\eta)\Delta < 0,$$

and η declines. The policy trade-off is clear: more public debt improves risk sharing and lowers risk premia up to the bound; push debt to (or beyond) the bound and risky claims vanish, removing the very margin that prices idiosyncratic risk. Returning to the interior ($\theta^f > 0$) then requires bringing b back below the bound (or shifting fundamentals so that V rises relative to b).

6.3 Bubble equilibrium with public debt

With $p_t > 0$, both the bubble and public bonds supply safe value. Define the net safe asset per unit capital

$$S_t \;\equiv\; (1-\underline{\chi})\, p_t \;+\; b_t. \tag{53}$$

Proposition 6 (Public debt under a positive bubble)**.** *Fix $\Delta > 0$ and $\eta \in (0,1)$. Let $V \equiv V(\eta)$, $\theta^b \equiv q^b/V$, and $S \equiv (1-\underline{\chi})p + b$.*

1. ***Net safe-asset identity and crowd-out (fixed η).***

$$q^b + S = V, \qquad \left.\frac{\partial q^b}{\partial b}\right|_\eta + \left.\frac{\partial S}{\partial b}\right|_\eta = 0. \tag{54}$$

Equivalently, $dq^b + dS = 0$ at fixed η: debt changes only the split between risky equity and net safe assets.

2. ***Bubble steady state (invariance and one-for-one crowd-out).*** *When $p > 0$, risk pricing pins exposure,*

$$\Gamma^b = \sqrt{\Delta} \quad\Longleftrightarrow\quad \theta^b_\infty = \frac{\sqrt{\Delta}\,\bar{\eta}}{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}} \in (0,1), \tag{55}$$

with

$$\bar{\mu}^\eta = 0, \qquad \bar{r}^f = \rho - \delta^b + g, \qquad \bar{S} = V(\bar{\eta})\big(1-\theta^b_\infty\big). \tag{56}$$

At steady state $\bar{\eta}$, $\bar{q}^b = \theta^b_\infty V(\bar{\eta})$ is locally invariant to $\bar{b}$ and

$$\left.\frac{\partial\left[(1-\underline{\chi})\,p\right]}{\partial b}\right|_{\bar{\eta}} = -1, \qquad \left.\frac{\partial q^b}{\partial b}\right|_{\bar{\eta}} = 0. \tag{57}$$

Thus additional debt crowds out the bubble one-for-one at the bubble steady state, leaving q^b and r^f unchanged.

3. ***Debt threshold for bubble existence.*** *At given (η, q^b),*

$$b^{\max}(\eta) = V(\eta) - q^b, \tag{58}$$

and if $b \geq b^{\max}(\eta)$ the bubble vanishes on impact ($p = 0$) and the economy reverts to the fundamental regime. At the bubble steady state,

$$\bar{b}^{\max}(\bar{\eta}) = V(\bar{\eta})\left(1-\theta^b_\infty\right) = \frac{a-s}{A(\bar{\eta})}\left[1 - \frac{\sqrt{\Delta}\,\bar{\eta}}{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}\right]. \tag{59}$$

4. ***Transition with fixed b: decomposition via (q, S).*** *Using $q^b = \theta^b V$ and $S = V - q^b = (1-\theta^b)V$,*

$$\begin{aligned} \frac{dq^b_t}{q^b_t} &= \mu^{q^b}_t\,dt = \mu^V_t\,dt + \mu^{\theta^b}_t\,dt, \qquad dS_t = (1-\theta^b_t)\,dV_t - V_t\,d\theta^b_t, \\ \frac{dV_t}{V_t} &= \mu^V_t\,dt = \frac{\delta^e\eta_t}{A(\eta_t)}\,\mu^\eta_t\,dt, \qquad r^f_t = \rho - \delta^b + g + B(\eta_t)\mu^\eta_t. \end{aligned} \tag{60}$$

Hence q^b and net safe supply S are driven by discount-rate revaluation of V and composition dynamics in θ^b. The bubble level follows from $p_t = \left(S_t - b_t\right)/(1-\underline{\chi})$.

Fixed-η accounting and composition. At a given wealth distribution η, total valuation $V(\eta)$ splits into risky capitalization q^b and net safe value $S \equiv (1-\underline{\chi})p + b$:

$$q^b + S = V(\eta).$$

This is pure accounting: it holds regardless of whether $p > 0$ or not.

Holding η fixed, a marginal bond issue db must be absorbed by a change in composition:

$$dq^b + dS = 0 \quad \Longrightarrow \quad \left.\frac{\partial q^b}{\partial b}\right|_\eta = -\left.\frac{\partial S}{\partial b}\right|_\eta.$$

By itself, this identity does not yet determine *who* adjusts (risky vs. bubble); the answer comes from risk pricing in the interior bubble equilibrium.

Invariance and one-for-one crowd-out. When $p > 0$, the equilibrium pins entrepreneurs' undiversified exposure at $\Gamma^b = \sqrt{\Delta}$. Equivalently,

$$\theta_\infty^b = \frac{\sqrt{\Delta}\,\bar{\eta}}{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}} \in (0,1),$$

so at a fixed $\bar{\eta}$ the risky share θ_∞^b is *locally invariant*.

With θ_∞^b fixed, $q^b = \theta_\infty^b V(\bar{\eta})$ is locally unchanged by b, hence

$$\left.\frac{\partial q^b}{\partial b}\right|_{\bar{\eta}} = 0, \qquad \left.\frac{\partial\left[(1-\underline{\chi})p\right]}{\partial b}\right|_{\bar{\eta}} = -1.$$

Government bonds and the bubble are thus *perfect substitutes in safe units*: an extra dollar of bonds crowds out one dollar of bubble safety. Because entrepreneurs' risk bearing is unchanged, the inequality drift remains shut down ($\bar{\mu}^\eta = 0$) and the safe rate stays at its anchor, $\bar{r}^f = \rho - \delta^b + g$.

Intuitively, savers' net demand for safety at $\bar{\eta}$ is $S = \left(1 - \theta_\infty^b\right)V(\bar{\eta})$. If the government supplies more of it via bonds, the market supplies less via the bubble, leaving the risky share—and hence entrepreneurs' undiversified exposure—unchanged.

Debt capacity and the crowd-out threshold. At a given (η, q^b), the largest debt consistent with $p \geq 0$ is

$$b^{\max}(\eta) = V(\eta) - q^b.$$

In the bubble interior (steady state),

$$\bar{b}^{\max}(\bar{\eta}) = \underbrace{V(\bar{\eta})}_{\text{total value}} \cdot \underbrace{(1-\theta^b_\infty)}_{\text{net safe share}} = \frac{a-s}{A(\bar{\eta})}\left[1 - \frac{\sqrt{\Delta}\,\bar{\eta}}{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}\right].$$

If $b \uparrow b^{\max}$, the bubble shrinks to $p \downarrow 0$ while q^b and Γ remain at their interior values. At the instant p hits zero, the economy switches to the fundamental regime with $\Gamma^f = \sqrt{\Delta}$, so r^f and q do not jump—the bubble "dies quietly."

Debt capacity rises with total value V (higher a, lower s, or lower $A(\eta)$) and falls when entrepreneurs must bear more risk in the interior (larger $\sqrt{\Delta}$ or smaller $(1-\tau)\underline{\chi}\tilde{\sigma}$). Intuitively, the more risk entrepreneurs must carry, the less room there is for safe assets (bonds plus bubble).

Dynamics with fixed b: valuation vs. composition. Write $q^b = \theta^b V$ and $S = (1-\theta^b)V$. Then

$$\frac{dq^b_t}{q^b_t} = \underbrace{\mu^V_t}_{\text{discount-rate revaluation}} dt + \underbrace{\mu^{\theta^b}_t}_{\text{composition}} dt, \qquad dS_t = (1-\theta^b_t)\,dV_t - V_t\,d\theta^b_t,$$

with

$$\frac{dV_t}{V_t} = \mu^V_t\,dt = \frac{\delta^e \eta_t}{A(\eta_t)}\mu^\eta_t\,dt, \qquad r^f_t = \rho - \delta^b + g + B(\eta_t)\mu^\eta_t.$$

In steady state, $\mu^\eta_t = 0$ and pricing pins θ^b, so $\mu^V_t = \mu^{\theta^b}_t = 0$: q^b and S are flat unless policy/parameters move.

Along transition path, both V (via η_t) and θ^b can move; the decomposition above isolates the discount-rate channel from the composition channel.

Contrast with the fundamental (no-bubble) regime. With $p = 0$, bonds cannot be offset by a market-created safe asset. A higher b directly lowers the risky share borne by entrepreneurs, weakens their precautionary saving, and—when $\delta^e < \rho$—raises r^f on impact. With $p > 0$, small changes in b are absorbed by the bubble; entrepreneurs' risk bearing, μ^η, and r^f are locally unchanged.

Policy experiments and comparative statics (at $p > 0$). Inside the bubble region, a small surprise debt issuance at fixed η simply replaces bubble-supplied safety with government bonds: $d[(1-\underline{\chi})p] = -db$, while risky capitalization is unchanged ($dq^b = 0$) and the safe rate does not move ($dr^f = 0$). A large issuance that lifts b to the capacity $b^{\max}$ fully crowds out the bubble; beyond that threshold, additional debt works through the fundamental channel by lowering entrepreneurs' risky share, slowing the inequality drift, and typically lifting r^f. Changes in τ, $\underline{\chi}$, or $\tilde{\sigma}$ act by shifting the limiting risky share θ^b_∞—and thus the net safe share $1 - \theta^b_\infty$ and the safe-asset capacity S: higher τ (i.e., lower $(1-\tau)$) raises θ^b_∞ and reduces S, which can crowd out the bubble at a given b, whereas higher $\underline{\chi}$ or $\tilde{\sigma}$ lowers θ^b_∞ and increases S, permitting a larger bubble (or more bonds) at the same η without leaving the interior.

Key takeaways. Public debt and the bubble are locally perfect substitutes. Holding η fixed, the safe-asset identity $(1-\underline{\chi})p + b = S(\eta)$ implies that a small increase in b is absorbed one-for-one by a decline in $(1-\underline{\chi})p$, leaving risky capitalization q^b, entrepreneurs' exposure and risk premia, the inequality drift μ^η, and the safe rate r^f unchanged to first order.

Debt capacity is finite: when b reaches $b^{\max}(\eta)$ the bubble is smoothly crowded out ($p \to 0$) and the economy reverts to the fundamental regime; beyond that point, additional debt works through the standard composition channel—reducing entrepreneurs' risky share, compressing risk premia, slowing inequality drift, and, when $\rho > \delta^e$, lifting r^f by weakening precautionary saving.

Along the transition path or under large policy moves that shift η—changes in q^b and in safe capacity S decompose into (i) a valuation component via $V(\eta)$, which revalues with the discount-rate state μ^η, and (ii) a composition component via the risky share θ^b; correspondingly, the safe rate tracks the precautionary wedge $B(\eta)\mu^\eta$, rising when improved risk sharing lowers μ^η (if $B(\eta) < 0$).

6.4 Transition analysis with public debt: what changes relative to no debt?

To isolate novel implications of public debt, fix fiscal instruments (τ, s) and compare paths at the same initial η.

Wealth inequality. In fundamental equilibrium, public debt lowers $\theta^f = 1 - b/V$ and thus Γ^f on impact, reducing μ^η. Inequality still rises if $\Gamma^f > \sqrt{\Delta}$, but more slowly than without debt; the steady state satisfies $(\Gamma^f)^2 = \Delta$ at a lower $\bar{\eta}^f(b)$.

In bubble equilibrium. at fixed η, debt lowers θ^b and Γ^b; however, as the bubble persists, pricing pushes $\Gamma^b \to \sqrt{\Delta}$, neutralizing the first-order impact of b on μ^η in the long run.

Asset prices (q and p). In fundamental equilibrium, total value $V(\eta)$ evolves via the discount-rate revaluation channel as η changes, with level $V = (a - s)/A(\eta)$. With fixed public debt level b, capital price $q^f = V - b$ co-moves with total valuation V, but less than one-for-one. On impact, an increase in public debt b reduces capital price q^f one-for-one.

In bubble equilibrium, fundamental capital price q^b and net safe asset value S split total valuation V. At the bubble steady state, fundamental capital price q^b is pinned by θ^b_∞ and additional debt crowds out the bubble one-for-one. Along the path with fixed b, p co-moves with $S = V(1 - \theta^b) - b$: it rises with discount-rate revaluation and with declines in θ^b.

Risk-free rate. In both regimes, $r^f_t = \rho - \delta^b + g + B(\eta_t)\mu^\eta_t$. Debt lowers μ^η at matched η, hence raises r^f relative to the no-debt path.

In fundamental equilibrium, the lift in r^f is present both on impact and along the transition, keeping r^f closer to its anchor.

In bubble equilibrium steady state, r^f is locally invariant to debt (since $\mu^\eta = 0$). Thus debt reallocation operates through the safe-asset composition (bubble vs. bonds), not through the safe rate.

Government debt dynamics. From (40),

$$db_t = \Big[(r^f_t - g)b_t + s - \tau\big(a + (g + \mu^q_t)q_t\big)\Big]\, dt.$$

At a stationary point ($\mu^\eta = \mu^q = 0$), $\bar{b}$ solves $(\rho - \delta^b)\bar{b} + s = \tau\big(a + g\,\bar{q}\big)$, with $\bar{q}$ given by Proposition 5 in the fundamental regime and by $\bar{q}^{\,b} = \theta^b_\infty V(\bar{\eta})$ in the bubble regime. During transitions, taxation of capital gains (the $\tau\,\mu^q_t q_t$ term) feeds back into db_t, which is quantitatively relevant when revaluations are steep.

Summary. Public debt supplies safe value that (i) crowds out risky equity one-for-one at fixed η in the fundamental regime, slowing wealth concentration and lifting the safe rate relative to the no-debt path; but (ii) crowds out the *bubble's* safe value one-for-one at fixed η in the bubble regime, leaving risky capitalization and the safe rate locally unchanged at the steady state. The net safe-asset value $S = (1 - \underline{\chi})p + b$ unifies these results and clarifies the bubble-collapse threshold.

7 Safe-Asset Expansions: Inequality, Asset Prices, and Rates

We study three conceptually distinct safe-asset expansions—(i) privately created via financial innovation, (ii) market-created via a valuation bubble, and (iii) publicly supplied via government debt—and characterize their effects on wealth inequality (entrepreneurs' wealth share), the price of risky capital, and the risk-free rate through a unified mechanism and pass-through map.

Unified objects and mechanism. Let

$$\Delta \equiv \delta^b - \delta^e > 0, \qquad A(\eta) \equiv \rho - \delta^e \eta > 0, \qquad V(\eta) \equiv \frac{a - s}{A(\eta)}.$$

Note that $s = 0$ in the absence of government. Let $\theta \in (0, 1]$ be the risky share of private financial wealth, and define the exposure wedge

$$\phi(\eta) \equiv (1 - \tau)\,\underline{\chi}\,\tilde{\sigma}/\eta.$$

Entrepreneurs' undiversified risk exposure per unit wealth and inequality drift are

$$\Gamma = \theta\,\phi(\eta), \qquad \mu^{\eta} = (1-\eta)\big(\Gamma^2 - \Delta\big).$$

Under log utility, Γ is both the quantity of undiversified risk borne by entrepreneurs and the price of idiosyncratic risk (a Sharpe ratio). The risk-free rate is

$$r^f = \rho - \delta^b + g + B(\eta)\,\mu^{\eta}, \qquad B(\eta) \equiv \frac{\delta^e\,\eta}{\rho - \delta^e \eta} - \frac{\eta}{1-\eta}.$$

Discounting pins total valuation per unit of capital: at a given η, $V(\eta)$ is *unchanged* by composition shifts between risky and safe assets.

Proposition 7 (Unified pass-through map at fixed η)**.** *Consider any change that reduces entrepreneurs' undiversified risk exposure Γ either by (i) lowering the risky share θ (pure composition) and/or (ii) lowering the exposure wedge $\phi(\eta)$ (technology/financing). Holding η fixed:*

1. *Risk bearing and inequality: $d(\Gamma^2) \le 0$, so $d\mu^{\eta} = (1-\eta)\,d(\Gamma^2) \le 0$. Inequality pressure eases and reverses if Γ^2 falls below Δ.*
2. *Asset values: $V(\eta)$ is unchanged. If the change acts via composition ($d\theta < 0$), then $dq = d(\theta V) = -V\,d\big[(1-\theta)\big] < 0$. If it acts purely via $\phi(\eta)$, then $dq = 0$ on impact.*
3. *Risk-free rate: $dr^f = B(\eta)\,(1-\eta)\,d(\Gamma^2)$. Hence r^f rises on impact when $\delta^e < \rho$ (so $B(\eta) < 0$), falls when $\delta^e > \rho$, and is locally unchanged when $\delta^e = \rho$.*
4. *Long run: As $\mu^{\eta} \to 0$, $r^f \to \rho - \delta^b + g$ in all cases.*

7.1 Mechanism 1: Financial Innovation (Private Safe Assets)

Setup. Financial innovation reduces the exposure wedge $\phi(\eta)$ through: safer projects (lower $\tilde{\sigma}$), looser equity constraint (lower $\underline{\chi}$). And this expands private safe-asset capacity. Because entrepreneurs can issue more private risk-free debt to traditional savers due to lower risk exposure and thus lower precautionary saving motives. In our baseline fundamental regime without public debt or bubbles, the risky share is $\theta^f = 1$; financial innovation is modeled as a *structural* de-risking ($\phi(\eta) \downarrow$) with no immediate composition shift (θ unchanged on impact).

Proposition 8 (Financial innovation: pass-through at fixed η)**.** *Holding η fixed and keeping θ constant on impact (e.g., $\theta^f = 1$ in fundamentals), the impact pass-through of a change in $\phi(\eta)$ is*

$$d(\Gamma^2) = \theta^2\, d\big(\phi(\eta)^2\big), \qquad d\mu^\eta = (1-\eta)\,\theta^2\, d\big(\phi(\eta)^2\big), \qquad dr^f = B(\eta)\,(1-\eta)\,\theta^2\, d\big(\phi(\eta)^2\big).$$

Equivalently, for primitives,

$$\frac{\partial(\Gamma^2)}{\partial \underline{\chi}} = \frac{2\,\theta^2\,\phi(\eta)^2}{\underline{\chi}}, \qquad \frac{\partial(\Gamma^2)}{\partial \tilde{\sigma}} = \frac{2\,\theta^2\,\phi(\eta)^2}{\tilde{\sigma}}.$$

On impact, $dq = 0$ (no composition shift), while r^f moves with the sign in Proposition 7.

Intuition. Financial innovation makes entrepreneurial cash flows inherently safer or better diversifiable. Because $V(\eta)$ is pinned by discounting and there is no immediate composition change, the price of risky capital does not jump. The entire impact operates through a lower exposure wedge $\phi(\eta)$, which reduces undiversified risk Γ, slows inequality drift, and—when $\delta^e < \rho$—raises the risk-free rate on impact. As $\mu^\eta \to 0$, $r^f \to \rho - \delta^b + g$.

7.2 Mechanism 2: Bubbles (Market-Driven Safe Assets)

Setup. A valuation bubble p is a safe, non-dividend component on public equity. The value identity reads

$$V(\eta) = q^b + (1-\underline{\chi})\,p + b,$$

with risky share $\theta^b \equiv q^b/V(\eta) \in (0,1]$ and exposure $\Gamma^b = \theta^b\,\phi(\eta)$.

Proposition 9 (Bubble as a market-created safe asset: pass-through)**.** *Fix η and b. Then*

$$\frac{\partial q^b}{\partial p} = -(1-\underline{\chi}), \qquad \frac{\partial \theta^b}{\partial p} = -\frac{1-\underline{\chi}}{V(\eta)}, \qquad \frac{\partial (\Gamma^{b\,2})}{\partial p} = -\frac{2\,(1-\underline{\chi})\,\theta^b}{V(\eta)}\,\phi(\eta)^2 < 0.$$

Consequently,

$$\frac{\partial \mu^\eta}{\partial p} = (1-\eta)\,\frac{\partial (\Gamma^{b\,2})}{\partial p}, \qquad \frac{\partial r^f}{\partial p} = B(\eta)\,(1-\eta)\,\frac{\partial (\Gamma^{b\,2})}{\partial p}.$$

When $\delta^e < \rho$ (so $B(\eta) < 0$), r^f rises with p on impact.

Intuition. A bubble is a *pure composition* device that inserts a safe, non-dividend component into aggregate equity, reducing θ and shifting value from risky to safe at fixed $V(\eta)$. Along transition paths with $p > 0$, Γ typically falls with θ, easing inequality drift. The proposition describes the local pass-through mechanism that operates along the transition path. In a bubbly steady state, which is the long-run equilibrium point where this mechanism brings the system to rest, no-arbitrage pins

$$\Gamma^b = \sqrt{\Delta}, \qquad \theta^b = \frac{\sqrt{\Delta}}{\phi(\eta)}, \qquad \mu^\eta = 0, \qquad r^f = \rho - \delta^b + g.$$

7.3 Mechanism 3: Public Debt (Government Safe Assets)

Setup. Government bonds b are safe assets publicly supplied. The exposure wedge $\phi(\eta) = (1-\tau)\,\underline{\chi}\,\tilde{\sigma}/\eta$ maps the risky share θ into undiversified risk exposure $\Gamma = \theta\,\phi(\eta)$. The value identity is

$$V(\eta) = q + (1-\underline{\chi})\,p + b.$$

Fundamental regime ($p = 0$)**.** With no bubble, public debt replaces risky private claims one-for-one:

$$q^f = V(\eta) - b, \qquad \theta^f = 1 - \frac{b}{V(\eta)}, \qquad \Gamma^f = \theta^f\,\phi(\eta).$$

Proposition 10 (Debt in fundamentals: composition, de-risking, and pass-through)**.** *Fix η and*

$p = 0$. *Then*

$$\frac{\partial q^f}{\partial b} = -1, \qquad \frac{\partial \theta^f}{\partial b} = -\frac{1}{V(\eta)}, \qquad \frac{\partial (\Gamma^{f\,2})}{\partial b} = -\frac{2\,\theta^f}{V(\eta)}\,\phi(\eta)^2 < 0, \qquad \frac{\partial^2 (\Gamma^{f\,2})}{\partial b^2} = \frac{2\,\phi(\eta)^2}{V(\eta)^2} > 0.$$

Hence

$$\frac{\partial \mu^\eta}{\partial b} = (1-\eta)\,\frac{\partial (\Gamma^{f\,2})}{\partial b}, \qquad \frac{\partial r^f}{\partial b} = B(\eta)\,(1-\eta)\,\frac{\partial (\Gamma^{f\,2})}{\partial b}.$$

When $\delta^e < \rho$ *(so* $B(\eta) < 0$*), a marginal increase in* b *raises* r^f *on impact. As* θ^f *shrinks with* b*, the marginal de-risking* $|\partial(\Gamma^{f\,2})/\partial b|$ *and rate pass-through attenuate.*

Bubbly regimes with public debt ($p > 0$). When a bubble is present, the safe component $(1-\underline{\chi})\,p$ and public debt b are close substitutes in the value identity,

$$q^b + (1-\underline{\chi})\,p + b = V(\eta), \qquad \theta^b = 1 - \frac{(1-\underline{\chi})\,p + b}{V(\eta)}, \qquad \Gamma^b = \theta^b\,\phi(\eta).$$

Proposition 11 (Debt in bubbles: absorption band and kinked pass-through)**.** *Fix* η *and consider date* t.

1. *Interior absorption (steady state): If* $p_t > 0$*, an unanticipated marginal issuance* db_t *is absorbed by the bubble at the steady state,*

$$d\big[(1-\underline{\chi})p_t\big] = -\,db_t, \quad dq_t^b = d\theta_t^b = d(\Gamma_t^{b\,2}) = dr_t^f = d\mu_t^\eta = 0.$$

2. *Finite absorption band: Define slack*

$$(1-\underline{\chi})p_t = b_t^{\max} - b_t, \qquad b_t^{\max} \equiv V(\eta_t) - q_t^b = V(\eta_t)\big(1-\theta_t^b\big).$$

   For any finite change $|\Delta b_t| \le s_t$*,*

$$\Delta((1-\underline{\chi})\,p_t) = -\Delta b_t, \qquad \Delta q_t^b = \Delta\theta_t^b = \Delta(\Gamma_t^{b\,2}) = \Delta r^f = \Delta\mu_t^\eta = 0.$$

3. *Boundary and kink: When $b_t \uparrow b_t^{\max}$ (equivalently $p_t \downarrow 0$), the economy exits the bubbly region. Levels are continuous in q and r^f, but the policy multiplier kinks:*

$$\frac{\partial(\Gamma^2)}{\partial b} = \begin{cases} 0, & p > 0, \\ -\dfrac{2\,\theta^f}{V(\eta)}\,\phi(\eta)^2 < 0, & p = 0, \end{cases}$$

$$\frac{\partial r^f}{\partial b} = \begin{cases} 0, & p > 0, \\ B(\eta)\,(1-\eta)\,\dfrac{\partial(\Gamma^{f\,2})}{\partial b}\ (\textit{sign as in Proposition 10}), & p = 0. \end{cases}$$

4. *Bubbly steady state: No-arbitrage pins*

$$\Gamma^b = \sqrt{\Delta}, \qquad \theta^b = \frac{\sqrt{\Delta}}{\phi(\eta)}, \qquad \mu^\eta = 0, \qquad r^f = \rho - \delta^b + g.$$

Intuition. In bubble equilibrium steady state, the bubble itself is a safe component. New public debt is a close substitute and is absorbed one-for-one by shrinking the bubble's value, leaving the risky share—and thus risk, inequality drift, and the risk-free rate—unchanged. Once the bubble is exhausted, fundamentals reassert: the levels of q and r^f are smooth at the boundary, but the policy multiplier kinks from zero to the strictly positive fundamental pass-through.

7.4 What is common and what is different

Common logic across the three expansions. All three mechanisms operate by lowering the sufficient statistic $\Gamma = \theta\,\phi(\eta)$—either via composition ($\theta \downarrow$) or via de-risking ($\phi(\eta) \downarrow$). At fixed η, a reduction in Γ

(i) slows or reverses the inequality drift $\mu^\eta = (1-\eta)(\Gamma^2 - \Delta)$;

(ii) lowers the price of idiosyncratic risk borne by entrepreneurs;

(iii) moves the safe rate through $dr^f = B(\eta)(1-\eta)\,d(\Gamma^2)$, raising r^f on impact when $\delta^e < \rho$.

None of these interventions increases $V(\eta)$ on impact—discounting pins $V(\eta)$—so expansions reallocate value between risky and safe claims rather than adding net present value at a given η. As $\mu^{\eta} \to 0$, $r^f \to \rho - \delta^b + g$ in all cases.

How they differ: instrument, composition, and pass-through. Financial innovation is structural de-risking: it lowers $\phi(\eta)$ with θ unchanged on impact, so $dq = 0$ and the entire pass-through runs through $d(\Gamma^2) = \theta^2\, d(\phi^2)$.

A bubble is pure composition locally: $dq/dp = -(1 - \underline{\chi})$ and $\partial(\Gamma^2)/\partial p = -(2\,\theta\,\phi^2/V)\,(1 - \underline{\chi})$.

Public debt is a policy-driven composition tool with state-dependent efficacy: in fundamentals, $dq/db = -1$ and $\partial(\Gamma^2)/\partial b = -(2\,\theta\,\phi^2/V)$; in a bubbly regime, issuance is absorbed one-for-one by the bubble within an absorption band, with zero contemporaneous pass-through until the boundary is hit, where the policy multiplier kinks to the fundamental mapping.

8 Welfare Analysis

Key result (entrepreneurial welfare trade-off). Safe-asset expansions need not harm entrepreneurs. Our unified sufficient-statistic approach shows that all mechanisms that expand safety—private innovation, valuation bubbles, or public debt—operate only through entrepreneurs' undiversified exposure $\Gamma(\eta)$. A marginal fall in Γ^2 triggers two opposing forces: it raises the risk-free rate r^f in general equilibrium by easing precautionary saving, and it directly compresses the private risk premium earned by entrepreneurs. Evaluated at a common wealth distribution η, the entrepreneur's flow-welfare change is

$$\Delta\Psi^e(\eta) = C(\eta)\,\Delta\Gamma^2(\eta), \qquad C(\eta) = \tfrac{1}{2} - \frac{(\rho - \delta^e)\eta}{\rho - \delta^e\eta}.$$

This delivers a single threshold

$$\eta^* = \frac{\rho}{\rho + (\rho - \delta^e)} = \frac{\rho}{2\rho - \delta^e} \in (1/2, 1),$$

i.e., the ratio of the traditional savers' discount rate to the *sum* of the savers' and entrepreneurs' discount rates. For any safer policy ($\Delta\Gamma^2 \leq 0$), entrepreneurs benefit if and only if $\eta > \eta^*$, while savers always benefit since

$$\Delta\Psi^s(\eta) = (1-\eta)\, B(\eta)\, \Delta\Gamma^2(\eta) > 0 \quad \text{when} \quad B(\eta) < 0.$$

The entrepreneurial welfare trade-off and its distributional threshold are the core contribution: they organize all safe-asset mechanisms through a single statistic and a single cutoff.

Intuition and economic mechanism. Entrepreneurs are long the risky asset but benchmarked to the safe rate. Making the environment safer raises r^f—a boon to everyone—yet shrinks the risk premium that entrepreneurs earn. When entrepreneurs are relatively poor (low η), the marginal value of the risk premium is high relative to the value of a higher benchmark, so the direct risk premium effect dominates and they prefer risk. When entrepreneurs are rich (high η), protecting a large base and enjoying a higher risk-free rate r^f dominate, so they prefer safety. The coefficient $C(\eta)$ formalizes this logic: it decreases with η and crosses zero at η^*. Expressing η^* as

$$\eta^* = \frac{\rho}{\rho + (\rho - \delta^e)}$$

makes transparent that the threshold is the savers' share of total "impatience" in the economy (savers' discount rate divided by the sum of savers' and entrepreneurs' discount rates). The flip in preferences occurs exactly when these impatience weights equalize the direct risk-premium channel and the safe-rate channel.

Mechanism-agnostic mapping and policy reading. Because the two channels operate solely through Γ^2, the result is independent of how safety is created. Scale de-risking (financial innovation) and composition de-risking (bubbles or public debt) both lower Γ^2 and therefore engage the same trade-off summarized by $C(\eta)$. The practical policy rule is correspondingly simple: determine how the mechanism moves Γ^2, and locate the economy relative to η^*. Since interior steady states pin down $\Gamma^2 = \Delta$ and $r^f = \rho - \delta^b + g$, the welfare consequences are transitional and distributional; they hinge on how the policy reshapes the path of $\{\Gamma_t^2, \eta_t\}$ and on which side of η^* entrepreneurs spend most of their time. This is precisely why safe-asset expansions can be entrepreneur-friendly once entrepreneurs are already rich, even though the same de-risking can hurt them early on.

8.1 Setup and the sufficient statistic

We compare alternative regimes at the same wealth distribution η. With log rules $c^{e,i} = \rho^e W^{e,i}$ and $c^{s,j} = \rho W^{s,j}$, where $\rho^e := \rho - \delta^e$ is entrepreneurs' discount rate,

$$V_t^e = \frac{1}{\rho^e}\log\rho^e + \frac{1}{\rho^e}\log W_t^{e,i} + \frac{1}{\rho^e}\mathbb{E}_t\left[\int_0^\infty e^{-\rho^e u}\,\Psi_{t+u}^e\,du\right],$$
$$V_t^s = \frac{1}{\rho}\log\rho + \frac{1}{\rho}\log W_t^{s,j} + \frac{1}{\rho}\mathbb{E}_t\left[\int_0^\infty e^{-\rho u}\,\Psi_{t+u}^s\,du\right],$$

with matched-state flow utilities

$$\Psi^e = (r^f - \rho^e) + \tfrac{1}{2}\Gamma^2, \qquad \Psi^s = r^f - \rho.$$

Entrepreneurs' undiversified exposure is

$$\Gamma(\eta) = \frac{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta}\,\theta(\eta), \qquad \theta(\eta) \in (0,1].$$

Inequality drift and the safe rate satisfy

$$\mu^\eta = (1-\eta)(\Gamma^2 - \Delta), \qquad r^f = \rho - \delta^b + g + B(\eta)\,\mu^\eta, \quad B(\eta) < 0,$$

so a change in exposure moves the safe rate via

$$dr^f = B(\eta)\,(1-\eta)\,d\Gamma^2.$$

At matched states, $d\Gamma^2$ affects entrepreneurs' premium directly by $+\frac{1}{2}\,d\Gamma^2$ and affects both groups through $dr^f = B(\eta)(1-\eta)\,d\Gamma^2$. This decomposition delivers a simple welfare map.

Proposition 12 (Unified welfare map (pointwise at matched η))**.** *Fix $\eta \in (0,1)$ and compare X vs. Y with exposures $\Gamma^X(\eta)$ and $\Gamma^Y(\eta)$. Let $\Delta\Gamma^2(\eta) := (\Gamma^Y)^2 - (\Gamma^X)^2$. Then*

$$\Delta\Psi^s(\eta) = (1-\eta)\,B(\eta)\,\Delta\Gamma^2(\eta), \qquad \Delta\Psi^e(\eta) = C(\eta)\,\Delta\Gamma^2(\eta),$$

with

$$C(\eta) = \frac{1}{2} - \underbrace{\frac{(\rho-\delta^e)\eta}{\rho-\delta^e\eta}}_{A(\eta)} = \frac{1}{2} - \frac{\rho^e\eta}{\rho^e\eta+\rho(1-\eta)}, \qquad \eta^* = \frac{\rho}{\rho+(\rho-\delta^e)} = \frac{\rho}{2\rho-\delta^e}.$$

Therefore, if $\Delta\Gamma^2(\eta) \le 0$ (safer), savers strictly gain pointwise, and entrepreneurs gain if and only if $\eta > \eta^*$. Lifetime welfare integrates these flow differences along the transition path induced by the policy.

The interpretation is straightforward. Safer environments compress the risk premium, which hurts entrepreneurs directly, but they also raise r^f through general-equilibrium feedback, which benefits everyone. Entrepreneurs' net valuation of safety is summarized by $C(\eta)$, which is decreasing in η: poorer entrepreneurs prefer risk because the premium matters more to them, while richer entrepreneurs prefer safety because protecting a larger base and enjoying a higher benchmark rate dominate. Writing η^* as $\rho/[\rho+(\rho-\delta^e)]$ underscores that the cutoff depends only on relative impatience—savers' discount rate versus entrepreneurs' discount rate—and is thus mechanism-agnostic.

8.2 Mechanism 1: Financial innovation (private safe assets)

Financial innovation relaxes issuance or pledgeability constraints or reduces idiosyncratic risk, acting as a scale de-risking:

$$\varphi^{\mathrm{FI}}(\eta) = \frac{(1-\tau_{\mathrm{new}})\underline{\chi}_{\mathrm{new}}\tilde{\sigma}_{\mathrm{new}}}{(1-\tau_{\mathrm{old}})\underline{\chi}_{\mathrm{old}}\tilde{\sigma}_{\mathrm{old}}} \leq 1, \qquad \theta^{\mathrm{FI}} = 1,$$

so that $\Gamma^{\mathrm{new}}(\eta) = \varphi^{\mathrm{FI}}(\eta)\,\Gamma^{\mathrm{old}}(\eta)$ with $\Delta\Gamma^2(\eta) \leq 0$ for a genuine innovation.

Proposition 13 (Financial innovation: welfare)**.** *If innovation lowers exposure so that $\Delta\Gamma^2(\eta) \leq 0$ (strict on a set of positive measure), then at matched states traditional savers gain, while entrepreneurs gain if and only if $\eta > \eta^*$. Any interior steady state satisfies $\Gamma^2 = \Delta$ and $r^f = \rho - \delta^b + g$; welfare differences arise along the transition.*

A pure scale change compresses the risk premium without reallocating valuation across assets. The increase in the safe rate benefits savers unambiguously. For entrepreneurs, the sign flips at η^*, reflecting the trade-off between a smaller premium and a higher benchmark rate.

8.3 Mechanism 2: Valuation bubbles (market-driven safe assets)

A bubble inserts a safe, non-dividend component into equity, shifting composition rather than scale:

$$q^b + (1-\underline{\chi})p = \frac{a}{A(\eta)}, \qquad \theta^b(\eta) = \frac{q^b}{a/A(\eta)} \in (0,1], \qquad \Gamma^b(\eta) = \theta^b(\eta)\,\frac{\underline{\chi}\tilde{\sigma}}{\eta}.$$

Bubble equilibrium steady state pins $\Gamma^b(\eta) = \sqrt{\Delta}$.

Proposition 14 (Bubble vs. fundamentals: welfare)**.** *Relative to fundamentals with $\theta^{f,0} = 1$, a bubble implements $\Gamma^b = \theta^b\,\Gamma^{f,0}$, so $\Delta\Gamma^2(\eta) \leq 0$. At matched states, traditional savers always gain and entrepreneurs gain if and only if $\eta > \eta^*$. The long-run safe rate equals $\rho - \delta^b + g$ in both regimes.*

De-risking occurs by reallocating a fixed valuation from risky to safe assets. Because the mechanism works only through Γ^2, the welfare trade-off is identical to that of financial innovation.

8.4 Mechanism 3: Public debt (government safe assets)

A. Fundamental equilibrium ($p = 0$)**: composition de-risking.** When no bubble is present, public bonds replace risky private assets according to

$$q(\eta) + b(\eta) = \frac{a-s}{A(\eta)}, \quad \theta^d(\eta) = \frac{q}{q+b} = 1 - \frac{A(\eta)}{a-s} b(\eta), \quad \Gamma^d(\eta) = \theta^d(\eta) \frac{(1-\tau)\underline{\chi}\tilde{\sigma}}{\eta}.$$

An increase in b lowers Γ^2 and raises r^f because it reduces precautionary saving motives and inequality drift:

$$\Delta\Gamma^2(\eta) \leq 0, \qquad \frac{\partial \mu^\eta}{\partial b} < 0, \qquad \frac{\partial r^f}{\partial b} = B(\eta) \frac{\partial \mu^\eta}{\partial b} > 0.$$

Proposition 15 (Debt in fundamental equilibrium: welfare)**.** *At matched η, traditional savers gain pointwise from higher r^f, and entrepreneurs gain if and only if $\eta > \eta^*$. The long-run safe rate satisfies $r^f = \rho - \delta^b + g$.*

The intuition mirrors the previous mechanisms: government debt supplies safety, reduces exposure, and raises the safe rate. Entrepreneurs' preference depends on whether the benefit of a higher benchmark outweighs the cost of a smaller risk premium, which is exactly what $C(\eta)$ evaluates. The fact that η^* equals $\rho / [\rho + (\rho - \delta^e)]$ highlights that this cutoff depends only on the groups' relative discounting and not on the specific policy instrument.

B. Bubble present ($p > 0$)**: local neutrality and regime switch.** When a bubble already supplies safety at the steady state,

$$q^b + (1-\underline{\chi})p + b = \frac{a-s}{A(\eta)}, \qquad \Gamma^b(\eta) = \sqrt{\Delta},$$

and no-arbitrage makes marginal public debt a one-for-one substitute for the bubble, so $(1-\underline{\chi})\,dp = -db$ and $\partial\Gamma^2/\partial b = \partial\mu^\eta/\partial b = \partial r^f/\partial b = 0$ inside the bubbly region. Debt changes welfare only when it is large enough to crowd out the bubble and revert the economy to fundamentals. The threshold

$$b^{\max}(\overline{\eta}) = \frac{a-s}{A(\overline{\eta})}\big(1-\theta^b(\overline{\eta})\big) = \frac{a-s}{A(\overline{\eta})}\left(1-\frac{\sqrt{\Delta}\,\overline{\eta}}{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}\right)$$

marks the switch where $p \to 0$. At the boundary, $\theta^f = \theta^b$, so q and r^f are continuous; once the switch occurs, welfare follows the fundamentals case.

8.5 Global and long-run perspective

Lifetime welfare equals the discounted integral of matched-state flow differences along the policy-induced transition path. For traditional savers, this is $(1-\eta_t)B(\eta_t)\,\Delta\Gamma_t^2$ at each date; for entrepreneurs, it is $C(\eta_t)\,\Delta\Gamma_t^2$. The entrepreneurial sign therefore depends on how often and how far the economy resides above rather than below the threshold $\eta^* = \rho/[\rho + (\rho - \delta^e)]$. Any interior steady state satisfies $\Gamma^2 = \Delta$ and $r^f = \rho - \delta^b + g$, which implies that safe-asset mechanisms matter primarily by reshaping the path of $\{\Gamma_t^2, \eta_t\}$, not the long-run safe rate itself. This path dependence is precisely what gives the result its distributional content: the same policy can hurt entrepreneurs early in development when $\eta < \eta^*$ and benefit them later when $\eta > \eta^*$.

8.6 Policy map

At matched states, the sufficient statistic for welfare is Γ^2. If a mechanism lowers Γ^2, traditional savers benefit, while entrepreneurs benefit only when they are rich enough, that is, when $\eta > \eta^* = \rho/[\rho + (\rho - \delta^e)]$. Innovation works through scale (compressing risk exposure), while debt and bubbles work through composition (reallocating valuation from risky to safe assets), yet the welfare logic is identical because both operate through Γ^2. When a bubble is present, marginal public debt is locally neutral until it is large enough to eliminate the bubble, at which point the economy reverts to fundamental equilibrium and the usual trade-off reappears. Taken together, these observations yield a simple policy reading: sign the policy's effect on Γ^2, locate the economy relative to η^*, and remember that long-run rates are pinned down while transitional distributional effects drive welfare.

9 Conclusion

This paper explains how rising asset prices can coincide with falling safe rates and changing inequality, and when they can, instead, deliver less inequality and higher safe rates. The key is the amount of undiversified risk that entrepreneurs are forced to bear. When entrepreneurs become wealthier because they save more and earn a premium for bearing risk, the economy discounts the future less, valuations rise, and precautionary motives depress the safe rate. Safe asset expansions work by reducing entreprenuers' undiversified risk exposure—either by reallocating value from risky to safe assets or by making risk-sharing easier. They slow wealth concentration, compress risk premia, and raise the safe rate on impact.

Two policy messages follow. First, the presence of a stable equity bubble changes the effectiveness of public debt: small changes in debt are locally neutral because they are absorbed by the bubble, with no movement in risky capitalization, inequality dynamics, or the safe rate; once debt is large enough to eliminate the bubble, additional issuance resumes its usual de-risking effects. Second, welfare hinges on distribution, not long-run rates. Safer environments always help traditional savers on impact, while entrepreneurs benefit only once their group is sufficiently wealthy; early in the transition they prefer risk, later they prefer safety.

In practice, the framework offers a simple diagnostic: to sign the effects of a policy or a market development, assess how it changes entrepreneurs' undiversified risk exposure and where the economy stands in terms of the entrepreneurial wealth share. Future work can bring this sufficient-statistic approach to richer preference structures, additional assets such as housing, optimal stabilization policy, and quantitative evaluation against micro and macro data.

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A Proofs of Propositions

A.1 A.0 Lemmas used repeatedly

Lemma 1 (Log consumption rules and homothetic value)**.** *Under log utility, the optimal consumption rules for any entrepreneur i and saver j are*

$$c_t^{e,i} = \rho^e W_t^{e,i}, \qquad c_t^{s,j} = \rho W_t^{s,j}, \qquad \rho^e := \rho - \delta^e,$$

and the indirect values are homothetic:

$$V^e(W) = \frac{1}{\rho^e}\log W + const, \qquad V^s(W) = \frac{1}{\rho}\log W + const.$$

Sketch. Guess $V(W) = A + B\log W$ in each HJB. FOCs give $c/W = \rho^e$ (entrepreneurs) and $c/W = \rho$ (savers). See Merton (1969) and Cox–Huang (1989).

Lemma 2 (Valuation identities from goods clearing)**.** *Goods market clearing implies $C_t^e + C_t^s = (a-s)K_t$, hence*

$$A(\eta_t)\left(q_t + (1-\underline{\chi})p_t + b_t\right) \;=\; a - s,$$

i.e.

$$\textit{(Fundamental)}\;\; p_t = 0 = b_t \Rightarrow A(\eta_t)\,q_t = a, \qquad \textit{(Bubble, no bonds)}\;\; b_t = 0 \Rightarrow A(\eta_t)\big(q_t + (1-\underline{\chi})p_t\big) = a,$$

$$\textit{(Public debt, no bubble)}\;\; p_t = 0 \Rightarrow A(\eta_t)\big(q_t + b_t\big) = a - s.$$

Proof. By lemma 1, $C_t^e + C_t^s = \rho^e W_t^e + \rho W_t^s = [\rho - \delta^e\eta_t]W_t = A(\eta_t)W_t$. Total private wealth per unit of capital equals the value of private claims: $W_t/K_t = q_t + (1-\underline{\chi})p_t + b_t$. Combine with $C^e + C^s = (a-s)K$.

Lemma 3 (Inequality dynamics)**.** *Let $\eta_t \equiv \frac{W_t^e}{W_t^e + W_t^s}$ be entrepreneurs' wealth share. Assume (i) no aggregate diffusion; (ii) entrepreneurs have log utility with constrained exposure summarized by Γ_t; (iii) savers are diversified and face a level wedge $\Pi_t^s = \delta^b$ in log-wealth drift. Define $\delta^e \equiv \rho - \rho^e$ and $\Delta \equiv \delta^b - \delta^e > 0$. Then*

$$\frac{d\eta_t}{\eta_t} \;=\; (1-\eta_t)\big(\Gamma_t^2 - \Delta\big)\,dt.$$

Proof. With finite-variation wealth processes under logarithmic utility[12],

$$\frac{\dot W^e}{W^e} = r^f - \rho^e + \Gamma^2, \qquad \frac{\dot W^s}{W^s} = r^f - \rho + \delta^b.$$

Thus $\dot R/R = \Gamma^2 - \Delta$ for $R = W^e/W^s$, and $\dot\eta = \eta(1-\eta)\dot R/R$ yields the claim.

Lemma 4 (Safe-rate decomposition)**.** *For any regime (fundamental, bubble, debt),*

$$r_t^f \;=\; \rho - \delta^b + g \;+\; B(\eta_t)\,\mu_t^\eta, \qquad B(\eta) \equiv \frac{\delta^e\eta}{\rho - \delta^e\eta} - \frac{\eta}{1-\eta} = -\frac{(\rho-\delta^e)\,\eta}{(1-\eta)(\rho-\delta^e\eta)} < 0 \;\; \textit{if} \;\; \rho > \delta^e.$$

[12] More detailed derivation starting from HJB equation is in Appendix B

Sketch. *Differentiate the valuation identity in lemma 2 and combine both groups' Euler equations. The wealth-distribution drift μ^η loads with coefficient $B(\eta)$. Details in the main text.*

Lemma 5 (Return and index decompositions)**.** *With no aggregate diffusion and q_t of finite variation,*

$$dr_t^{k,i} = \Big(\frac{a}{q_t} + \mu_t^q + g\Big)dt + \tilde{\sigma}\, d\tilde{Z}_{i,t}, \qquad \mathbb{E}_t[dr_t^{k,i}] = \frac{a}{q_t} + \mu_t^q + g.$$

Public equity carries no idiosyncratic risk and satisfies

$$\mathbb{E}_t[dr_t^{mf}] = r_t^f + (1-\eta_t)\delta^b.$$

Let the outside-equity index weights be $\omega_t = q_t/(q_t+p_t)$ and $1-\omega_t = p_t/(q_t+p_t)$. Then

$$\mathbb{E}_t[dr_t^{mf}] = \omega_t\,\mathbb{E}_t[dr_t^{oe}] + (1-\omega_t)\,\mathbb{E}_t\Big[\frac{dP_t}{P_t}\Big] = \frac{a}{q_t+p_t} + g \quad \textit{whenever } \mu_t^q = \mu_t^p = 0.$$

A.2 Fundamental equilibrium

Proof of Proposition 1 (Fundamental steady state). With $p = 0 = b$, lemma 2 implies $A(\eta)q = a$. lemma 3 gives $\mu^\eta = (1-\eta)\big((\Gamma^f)^2 - \Delta\big)$, where $\Gamma^f = \phi(\eta)$ since $\theta = 1$. A steady state requires $\mu^\eta = 0$, hence $(\Gamma^f)^2 = \Delta$:

$$\left(\frac{\underline{\chi}\,\tilde{\sigma}}{\eta}\right)^2 = \Delta \quad\Longrightarrow\quad \overline{\eta}^f = \frac{\underline{\chi}\,\tilde{\sigma}}{\sqrt{\Delta}}.$$

Then $\overline{q}^f = a/(\rho - \delta^e\overline{\eta}^f)$ by lemma 2. Interiority requires $0 < \overline{\eta}^f < \min\{1, \rho/\delta^e\}$. With $\mu^\eta = 0$, lemma 4 yields $\overline{r}^f = \rho - \delta^b + g$. ■

Proof of Proposition 2 (Fundamental transition). Define $F(\eta) \equiv (1-\eta)\big((\phi(\eta))^2 - \Delta\big)$ with $\phi(\eta) = \underline{\chi}\tilde{\sigma}/\eta$. If $0 < \eta < \overline{\eta}^f$ then $(\phi(\eta))^2 > \Delta$, so $\mu^\eta = F(\eta) > 0$ and $\eta_t \uparrow \overline{\eta}^f$.

Since $q_t = a/A(\eta_t)$ and $A'(\eta) = -\delta^e < 0$, q_t rises along the transition. Differentiating $\ln q_t$ gives $d\ln q_t = (\delta^e\eta_t/A(\eta_t))\,\mu_t^\eta\,dt > 0$ while $\mu_t^\eta > 0$.

From lemma 4, $r_t^f = \rho - \delta^b + g + B(\eta_t)\mu_t^\eta$. For $\rho > \delta^e$ we have $B(\eta) < 0$, hence $r_t^f < \rho - \delta^b + g$ until convergence and $r_t^f \to \rho - \delta^b + g$ as $\mu_t^\eta \to 0$. ■

A.3 Bubble equilibrium

Proof of Proposition 3 (Bubble steady state). With $p > 0$, $b = 0$, lemma 2 gives $q + (1-\underline{\chi})p = a/A(\eta)$. Let $\theta \equiv q/[q + (1-\underline{\chi})p] = q/V(\eta) \in (0,1)$. lemma 3 gives $\mu^\eta = (1-\eta)(\Gamma^2 - \Delta)$ with $\Gamma = \theta\,\phi(\eta)$. A steady state requires $\Gamma^2 = \Delta$, so

$$\theta = \frac{\sqrt{\Delta}}{\phi(\eta)} = \frac{\sqrt{\Delta}\,\eta}{\underline{\chi}\,\tilde{\sigma}} \in (0,1).$$

Hence

$$\overline{q}^b = \theta\,V(\eta) = \frac{a}{A(\eta)}\cdot\frac{\sqrt{\Delta}\,\eta}{\underline{\chi}\tilde{\sigma}}, \qquad \overline{p}^b = \frac{1}{1-\underline{\chi}}\Big(\frac{a}{A(\eta)} - \overline{q}^b\Big).$$

To pin down $\overline{\eta}^b$, impose the index pricing $\mathbb{E}[dr^{mf}] = r^f + (1-\eta)\delta^b$ and the balanced-growth condition $\mathbb{E}[dP/P] = g$ (bubble stationary per unit of K, that is $\mu_t^p = \mu_t^q = 0$). From lemma 5, at steady state $\mathbb{E}[dr^{mf}] = a/(q+p) + g$. With $r^f = \rho - \delta^b + g$ on a steady state and rearranging,

$$\frac{a}{q+p} + g = \rho - \eta\delta^b + g \quad \Longleftrightarrow \quad q + p = \frac{a}{\rho - \eta\delta^b}.$$

Combining $q + (1-\underline{\chi})p = a/A(\eta)$ with $q + p = a/(\rho - \eta\delta^b)$ and eliminating (q, p) yields the quadratic

$$\alpha\,\eta^2 + \beta\,\eta + \gamma = 0, \quad \text{with} \quad \alpha = \frac{\delta^b\sqrt{\Delta}}{\tilde{\sigma}}, \quad \beta = -\delta^b - \rho\,\frac{\sqrt{\Delta}}{\tilde{\sigma}} + (1-\underline{\chi})\delta^e, \quad \gamma = \underline{\chi}\,\rho.$$

The admissible root in $(0, \min\{1, \rho/\delta^e\})$ is $\overline{\eta}^b$. Finally, since $\mu^\eta = 0$, lemma 4 gives $\overline{r}^f = \rho - \delta^b + g$. ■

Proof of Proposition 4 (Bubble transition). Write $V(\eta) = a/A(\eta)$ and $q^b = \theta^b V$, $S = (1-\underline{\chi})p = V - q^b$. Then

$$\frac{dq^b}{q^b} = \frac{dV}{V} + \frac{d\theta^b}{\theta^b}, \qquad dS = (1-\theta^b)\,dV - V\,d\theta^b,$$

with $dV/V = (\delta^e\eta/A(\eta))\,\mu^\eta\,dt$ by lemma 3. From lemma 4, $r_t^{f,b} = \rho - \delta^b + g + B(\eta_t)\mu_t^\eta$. Since $\Gamma^b = \theta^b\phi(\eta)$ and $\theta^b \in (0,1)$, at matched η the bubble path features lower Γ than fundamentals and hence a less negative wedge $B(\eta)\mu^\eta$; this yields a higher r_t^f than fundamentals at the same η. ■

A.4 Public debt

Proof of Proposition 5 (Debt in fundamental equilibrium). (1) With $p = 0$, lemma 2 yields $q + b = V(\eta)$.

(2) $\theta^f = q/V = 1 - b/V$, so $\Gamma^f = \theta^f\phi(\eta)$. Then $\mu^\eta = (1-\eta)[(\Gamma^f)^2 - \Delta]$; at fixed η, higher b lowers θ^f and μ^η.

(3) From lemma 4, $dr^f = B(\eta)\,d\mu^\eta$ at fixed η. With $\rho > \delta^e$, $B(\eta) < 0$, and $\partial\mu^\eta/\partial b < 0$, so $\partial r^f/\partial b > 0$.

(4) With fixed b, $q = V - b$ implies $dq/q = (V/(V-b))\,dV/V$, and $dV/V = (\delta^e\eta/A(\eta))\,\mu^\eta\,dt$; r^f follows lemma 4.

(5) In steady state, $(\Gamma^f)^2 = \Delta$ with $\Gamma^f = \frac{(1-\tau)\underline{\chi}\tilde{\sigma}}{\eta}(1 - b/V(\eta))$. Let $c := \frac{(1-\tau)\underline{\chi}\tilde{\sigma}}{\sqrt{\Delta}} > 0$. Solving

$$\sqrt{\Delta} = \frac{(1-\tau)\underline{\chi}\tilde{\sigma}}{\eta}\left(1 - \frac{b\,A(\eta)}{a-s}\right)$$

for η yields

$$\bar{\eta}^f(b) = \frac{c\left(1 - \frac{\rho b}{a-s}\right)}{1 - \frac{c\,\delta^e}{a-s}\,b},$$

provided $b < \min\{(a-s)/\rho,\ (a-s)/(c\,\delta^e)\}$ to ensure interiority. Then $\bar{q}^f = \bar{V} - b$ with $\bar{V} = (a-s)/(\rho - \delta^e\bar{\eta}^f)$, and $\bar{r}^f = \rho - \delta^b + g$. ■

Proof of Proposition 6 (Debt with bubble). (1) At fixed η, lemma 2 implies $q^b + S = V(\eta)$ with $S = (1-\underline{\chi})p + b$. Differentiating at fixed η gives $dq^b + dS = 0$.

(2) At an interior bubble steady state, $\Gamma^b = \sqrt{\Delta}$ pins $\theta^b_\infty = \sqrt{\Delta}/\phi(\eta) = \sqrt{\Delta}\,\eta/[(1-\tau)\underline{\chi}\tilde{\sigma}]$, which is locally invariant to b at fixed η. Hence $q^b = \theta^b_\infty V(\eta)$ is locally invariant, implying $d\big[(1-\underline{\chi})p\big] = -db$. Since $\mu^\eta = 0$, $\bar{r}^f = \rho - \delta^b + g$.

(3) The bubble exists iff $p \geq 0$, i.e. $S \geq b$. Since $S = V - q^b$, the maximal debt consistent with $p \geq 0$ at η is $b^{\max}(\eta) = V(\eta) - q^b$. Evaluated at the steady state,

$$\bar{b}^{\max}(\bar{\eta}) = V(\bar{\eta})\big(1-\theta^b_\infty\big) = \frac{a-s}{A(\bar{\eta})}\left[1 - \frac{\sqrt{\Delta}\,\bar{\eta}}{(1-\tau)\underline{\chi}\tilde{\sigma}}\right].$$

(4) With fixed b, write $q^b = \theta^b V$ and $S = (1-\theta^b)V$. Then

$$\frac{dq^b}{q^b} = \frac{dV}{V} + \frac{d\theta^b}{\theta^b}, \qquad dS = (1-\theta^b)\,dV - V\,d\theta^b.$$

Since $dV/V = (\delta^e\eta/A(\eta))\,\mu^\eta\,dt$ and $r^f = \rho - \delta^b + g + B(\eta)\mu^\eta$, the stated decomposition follows.

(5) Inside the bubbly region, a marginal db is absorbed by $d\big[(1-\underline{\chi})p\big] = -db$, so $dq^b = d\theta^b = d(\Gamma^{b\,2}) = d\mu^\eta = dr^f = 0$. At the boundary $p \to 0$, the economy reverts to fundamentals; levels are continuous, but the policy multiplier kinks from 0 to the fundamental pass-through. ■

Proposition 16 (Law of motion for public debt per unit of capital)**.** *Let B be public debt and K the capital stock with $dK/K = g\,dt$. The flow budget is*

$$dB = r^f B\,dt + sK\,dt - \tau\left(\frac{a}{q} + g + \mu^q\right) qK\,dt.$$

Then per unit capital $b = B/K$ evolves as

$$db = \Big[(r^f - g)\,b + s - \tau\big(a + (g+\mu^q)q\big)\Big]\,dt. \tag{61}$$

Proof. Using $db = d(B/K) = K^{-1}dB - (B/K)\,dK/K$ with $dK/K = g\,dt$ delivers (61). ■

A.5 Unified pass-through maps

Proof of Proposition 7 (Unified pass-through at fixed η). At fixed η, $V(\eta)$ is invariant. Any safer change lowers $\Gamma^2 = \theta^2\phi(\eta)^2$ either via composition ($\theta\downarrow$) or scale de-risking ($\phi(\eta)\downarrow$). Hence $d\mu^\eta = (1-\eta)\,d(\Gamma^2) \leq 0$. From lemma 4, $dr^f = B(\eta)(1-\eta)\,d(\Gamma^2)$; since $B(\eta) < 0$ for $\rho > \delta^e$, safer changes raise r^f. Asset values: $dV = 0$ at fixed η, so $dq = d(\theta V) = V\,d\theta$. If the change acts only via $\phi(\eta)$, then $d\theta = 0$ on impact and $dq = 0$. ■

Proof of Proposition 8 (Financial innovation). At fixed η and θ, $\Gamma^2 = \theta^2\phi(\eta)^2$ so $d(\Gamma^2) = \theta^2\,d(\phi(\eta)^2)$. Differentiating $\phi(\eta) = (1-\tau)\underline{\chi}\tilde{\sigma}/\eta$ gives the primitives' pass-through. lemma 4 delivers $dr^f = B(\eta)(1-\eta)\,d(\Gamma^2)$. Since θ is unchanged on impact, $dq = 0$. Other results and claims follow from directly apply the result from Proposition 7. ■

Proof of Proposition 9 (Bubble as composition). With V fixed, $q^b = \theta V$ and $p = \big(V - q^b\big)/(1-\underline{\chi})$, so $\partial q^b/\partial p = -(1-\underline{\chi})$ and $\partial\theta/\partial p = -(1-\underline{\chi})/V$. Then

$$\frac{\partial(\Gamma^{b\,2})}{\partial p} = 2\theta\,\phi(\eta)^2\,\frac{\partial\theta}{\partial p} = -\,\frac{2(1-\underline{\chi})\,\theta}{V}\,\phi(\eta)^2 < 0.$$

Apply lemma 4. ■

Proof of Proposition 10 (Debt in fundamentals). With V fixed and $p = 0$, $q^f = V - b$ so $\partial q^f/\partial b = -1$ and $\partial\theta^f/\partial b = -1/V$. Then $\partial(\Gamma^{f\,2})/\partial b = 2\theta^f\phi(\eta)^2\,\partial\theta^f/\partial b = -2\theta^f\phi(\eta)^2/V < 0$. lemma 4 gives the safe-rate pass-through; convexity follows since the first derivative of Γ^{f2} is linear in θ^f, and θ^f is itself linear in b. ■

Proof of Proposition 11 (Debt in bubbles). Inside the bubbly region with $p > 0$, $V = S + q^b$ is locally invariant at fixed η. Because $S = (1-\underline{\chi})p + b$, $d\big[(1-\underline{\chi})p\big] = -db$. Thus $\partial(\Gamma^2)/\partial b = 0$, and lemma 4 yields $\partial r^f/\partial b = 0$. The absorption band and the boundary kink follow from $p \geq 0$ and $q^b + S = V$. ■

A.6 Unified welfare map

Proof of Proposition 12 (Welfare). Under log utility (lemma 1), each group's HJB can be written as a flow term plus a continuation value. At a matched state η,

$$\Psi^s(\eta) = r^f - \rho, \qquad \Psi^e(\eta) = (r^f - \rho^e) + \sup_x\{x\Gamma - \tfrac{1}{2}x^2\} = (r^f - \rho^e) + \tfrac{1}{2}\Gamma^2.$$

At matched η, $dr^f = B(\eta)(1-\eta)\,d\Gamma^2$ (lemma 4). Therefore

$$d\Psi^s(\eta) = B(\eta)(1-\eta)\,d\Gamma^2, \qquad d\Psi^e(\eta) = B(\eta)(1-\eta)\,d\Gamma^2 + \tfrac{1}{2}\,d\Gamma^2 = \left(\tfrac{1}{2} - \frac{(\rho-\delta^e)\eta}{\rho-\delta^e\eta}\right)d\Gamma^2.$$

The entrepreneurial flow threshold at which $d\Psi^e = 0$ solves $\eta^* = \rho/(2\rho - \delta^e) \in (1/2, 1)$ if $\rho > \delta^e$. Since interior steady states satisfy $\Gamma^2 = \Delta$ and $r^f = \rho - \delta^b + g$, long-run welfare is equalized; all differences are transitional and distributional. ■

References for Appendix A. Merton, R. (1969), "Lifetime Portfolio Selection under Uncertainty," Review of Economics and Statistics; Cox, J., and C. Huang (1989), "Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process," Journal of Economic Theory.

B Full model derivation

B.1 HJB equations and FOCs

The HJB equation for entrepreneurs' problem (referenced as (3) in the main text) is

$$\rho^e V^{e,i}(W^{e,i}) = \max_{\{c_t^{e,i},\theta_t^{k,i},\theta_t^{oe,i},\theta_t^{mf}\}} \Big\{ \log c^{e,i} + V'(W^{e,i})\, W^{e,i}\, \mu_t^{w,e,i} + \frac{1}{2} V''(W^{e,i}) \big(W^{e,i} \tilde{\pi}^{w,e,i}\big)^2 + \lambda_t^i \big[(1-\underline{\chi})\theta_t^{k,i} + \theta_t^{oe,i}\big] \Big\},$$

where

$$\begin{aligned}
\frac{dW^{e,i}}{W^{e,i}} &= \mu_t^{w,e,i}\, dt + \tilde{\pi}^{w,e,i}\, d\tilde{Z}_t^i, \\
\mu_t^{w,e,i} &= -\frac{c^{e,i}}{W^{e,i}} + r_t^f + \theta_t^{k,i} \frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f\, dt\right]}{dt} + \theta_t^{oe,i} \frac{\mathbb{E}\left[dr_t^{oe,i} - r_t^f\, dt\right]}{dt} + \theta_t^{mf} \frac{\mathbb{E}\left[dr_t^{mf} - r_t^f\, dt\right]}{dt} \\
&\quad - (1-\eta_t)\big(\theta_t^{k,i} + \theta_t^{oe,i} + \theta_t^{mf,i}\big)\delta^b, \\
\tilde{\pi}^{w,e,i} &= \big(\theta_t^{k,i} + \theta_t^{oe,i}\big)\tilde{\sigma}.
\end{aligned}$$

Guess a value function $V^{e,i}(W^{e,i}) = \gamma_t + \rho^e \log W_t^{e,i}$ and take first-order conditions, we have

$$c_t^{e,i} = \rho^e W_t^{e,i}, \tag{62}$$

$$\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f\, dt\right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 + (1-\eta_t)\delta^b - \lambda_t^i(1-\underline{\chi}), \tag{63}$$

$$\frac{\mathbb{E}\left[dr_t^{oe,i} - r_t^f\, dt\right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 + (1-\eta_t)\delta^b - \lambda_t^i, \tag{64}$$

$$\frac{\mathbb{E}\left[dr_t^{mf} - r_t^f\, dt\right]}{dt} = (1-\eta_t)\delta^b. \tag{65}$$

The HJB equation for traditional savers' problem (referenced as (4) in the main text) is

$$\rho V^{s,j}(W^{s,j}) = \max_{\{c^{s,j}\}} \left\{ \log c^{s,j} + V'(W^{s,j})\, W^{s,j}\, \mu_t^{w,s,j} \right\},$$

where

$$\frac{dW^{s,j}}{W^{s,j}} = \mu_t^{w,s,j}\, dt, \tag{66}$$

$$\mu_t^{w,s,j} = -\frac{c^{s,j}}{W^{s,j}} + r_t^f + \alpha_t^{mf} \frac{\mathbb{E}[dr_t^{mf} - r_t^f\, dt]}{dt} - (1-\eta_t)\alpha_t^{mf}\delta^b. \tag{67}$$

Guess a value function $V^{s,j}(W^{s,j}) = \beta_t + \rho \log W_t^{s,j}$ and take first order conditions, we have

$$c_t^{s,j} = \rho W_t^{s,j}. \tag{68}$$

Note that savers do not carry any idiosyncratic risks because the stock market index fund diversifies idiosyncratic risks.

B.2 Fundamental equilibrium

Market clearing conditions for the fundamental economy:

$$aK_t = C_t^e + C_t^s = \rho^e W_t^e + \rho W_t^s = \left(\rho^e \eta_t^f + \rho(1-\eta_t^f)\right) q_t K_t, \tag{69}$$

$$V_t^{mf} = V_t^{oe}. \tag{70}$$

Public equity is exactly the pooled outside equities issued by entrepreneurs, so we have the return of public equity equals the expected return of outside equity,

$$dr_t^{mf} = \mathbb{E}_t\left[dr_t^{oe,i}\right]. \tag{71}$$

And since the public equity does not carry idiosyncratic risks, the return of public equity also equals the risk-free rate (adjusted for demographics) in equilibrium,

$$\mathbb{E}_t\left[dr_t^{oe,i}\right] = dr_t^{mf} = r_t^f + (1-\eta_t)\delta^b\, dt. \tag{72}$$

From entrepreneurs' first-order conditions, we have

$$\lambda_t^i = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 > 0,$$

so the equity constraint binds, we have

$$\theta_t^{k,i} + \theta_t^{oe,i} = \underline{\chi}\,\theta_t^{k,i}. \tag{73}$$

Steady state of fundamental economy

In the fundamental economy, the total wealth in the economy is $q_t K_t$. The value of bubble $P_t = 0$. Define entrepreneurs' wealth share as $\eta_t = \frac{W_t^e}{W_t^e + W_t^s} = \frac{W_t^e}{q_t K_t}$.

From first-order conditions, we have the asset pricing equation for capital

$$\frac{\mathbb{E}[dr_t^{k,i} - r_t^f\, dt]}{dt} = \frac{(\underline{\chi}\tilde{\sigma})^2}{\eta_t^f} + (1-\eta_t)\delta^b. \tag{74}$$

In equilibrium we also have

$$dr_t^{mf} = \mathbb{E}_t\left[dr_t^{oe,i}\right] = r_t^f\, dt + (1-\eta_t)\delta^b\, dt. \tag{75}$$

Optimal consumption ratios with logarithmic utilities: $\frac{C_t^e}{W_t^e} = \rho^e$ and $\frac{C_t^s}{W_t^s} = \rho$.

For each traditional saver j,

$$c_t^{s,j} = \rho\, W_t^{s,j}, \qquad \frac{dW_t^{s,j}}{W_t^{s,j}} = (r_t^f - \rho)\, dt.$$

Aggregating over all living savers (including newborns),

$$C_t^s = \sum_j c_t^{s,j} = \rho\, W_t^s \quad \Longrightarrow \quad \frac{C_t^s}{W_t^s} = \rho.$$

With wealth-carrying births at rate δ^b, sector wealth satisfies

$$\frac{dW_t^s}{W_t^s} = (r_t^f - \rho + \delta^b)\, dt.$$

We can now derive the evolution of entrepreneurs' wealth share,

$$\frac{d\eta_t^f}{\eta_t^f} = \underbrace{(1-\eta_t^f)\left(-(\delta^b - \delta^e) + \left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2\right)}_{\equiv \mu_t^{\eta,f}} dt. \tag{76}$$

Consumption goods market-clearing condition

$$aK_t = C_t^e + C_t^s = \rho^e W_t^e + \rho W_t^s = (\rho^e \eta_t^f + \rho(1-\eta_t^f))\, q_t K_t, \tag{77}$$

from which we can solve for the capital price q_t as a function of entrepreneurs' wealth share η_t^f,

$$q_t^f = \frac{a}{\rho - \delta^e \eta_t^f}. \tag{78}$$

The risk-free rate is given by

$$r_t^f = \rho + \mu_t^{c,s,j,f} = \rho^e + \mu_t^{c,e,i,f} - \underbrace{\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2}_{\text{precautionary saving motive}}, \tag{79}$$

where $\mu_t^{c,s,f}$ and $\mu_t^{c,e,f}$ are the growth rates of savers' and entrepreneurs' consumption, respectively.

Since in equilibrium we have $C_t^e = \rho^e \eta_t^f q_t K_t$ and $C_t^s = \rho(1-\eta_t^f) q_t K_t$, we have

$$\frac{dc_t^{e,i}}{c_t^{e,i}} = \mu_t^{c,e,f}\,dt + \tilde{\pi}^{c,e,i}\,d\tilde{Z}_t^i = \left(\mu_t^{\eta,f} + \mu_t^q + g\right)dt + \tilde{\pi}^{c,e,i}\,d\tilde{Z}_t^i, \tag{80}$$

$$\frac{dc_t^{s,j}}{c_t^{s,j}} = \mu_t^{c,s,f}\,dt = \left(-\delta^b - \frac{\eta_t^f}{(1-\eta_t^f)}\mu_t^{\eta,f} + \mu_t^q + g\right)dt. \tag{81}$$

Solving for the steady state, that is when $\mu_t^{\eta,f} = 0$ and $\mu_t^q = 0$. Combining (74) and (77), we have

$$\overline{q}^f = \frac{a}{\rho - \delta^e \overline{\eta}^f}, \tag{82}$$

$$\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^b - \delta^e}}, \tag{83}$$

$$\overline{p}^f = 0. \tag{84}$$

Here we focus on a non-degenerate steady-state wealth distribution and require that

$$\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^b - \delta^e}} < 1. \tag{85}$$

And at the steady state we have

$$\overline{r}^f = \rho - \delta^b + g, \tag{86}$$

and we also have the risk premium of capital at the steady state

$$\frac{\mathbb{E}[d\overline{r}^{k,i} - \overline{r}^f\,dt]}{dt} = \frac{(\underline{\chi}\tilde{\sigma})^2}{\overline{\eta}^f} + (1-\overline{\eta}^f)\delta^b = \underline{\chi}\tilde{\sigma}\sqrt{\delta^b - \delta^e} + \left(1 - \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^b - \delta^e}}\right)\delta^b. \tag{87}$$

B.3 Bubble equilibrium

Intuition for bubble

The value of the stock market index fund is

$$V_t^{mf} = \underbrace{(1-\chi_t) q_t K_t}_{\text{Value of outside equities, } V_t^{oe}} + \underbrace{P_t}_{\text{Bubble}}, \tag{88}$$

where $1-\chi_t$ is the fraction of capital issued as outside equity in the aggregate. Since entrepreneurs are identical before idiosyncratic risk is realized, we have $\chi_{it} = \chi_t$. The value of outside equities is $V_t^{oe} = (1-\chi_t) q_t K_t$ and the value of bubble is P_t.

Since $\chi_t = \chi_{it} \geq \underline{\chi}$, we have

$$V_t^{oe} \leq (1-\underline{\chi}) q_t K_t.$$

If there is no bubble, we simply have outside equity market clearing as

$$V_t^{oe} = V_t^{mf}.$$

The value of the stock market index fund held by entrepreneurs is also constrained as

$$V_t^{oe} = V_t^{mf} \leq (1 - \underline{\chi}) q_t K_t. \tag{89}$$

A bubble with positive value relaxes this constraint for the stock market index fund because now we have

$$V_t^{mf} = V_t^{oe} + P_t,$$

and the constraint (89) becomes

$$V_t^{mf} \leq (1 - \underline{\chi}) q_t K_t + P_t. \tag{90}$$

Bubbles relax the indirect limit on public equity due to the skin-in-the-game constraint.

Bubble equilibrium

Market clearing conditions for the bubble economy:

$$aK_t = (\rho^e \eta_t^b + \rho(1 - \eta_t^b))(q_t K_t + P_t), \tag{91}$$

$$V_t^{mf} = V_t^{oe} + P_t. \tag{92}$$

Since $V_t^{mf} = V_t^{oe} + P_t$, the public equity is the pooled outside equity plus the bubble; the return of public equity is now

$$\mathbb{E}_t \left[dr_t^{mf} \right] = \frac{V_t^{oe}}{V_t^{mf}} \mathbb{E}_t \left[dr_t^{oe,i} \right] + \frac{P_t}{V_t^{mf}} \mathbb{E}_t \left[\frac{dP_t}{P_t} \right]. \tag{93}$$

The public equity does not carry any idiosyncratic risk, so we have in equilibrium

$$dr_t^{mf} = r_t^f \, dt + (1 - \eta_t) \delta^b \, dt. \tag{94}$$

We need to use other equilibrium conditions to determine the return of outside equity and the amount of outside equity issuance. In equilibrium, we have the return of outside equity equals the return of inside equity,

$$\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f \, dt \right]}{dt} = \frac{\mathbb{E}\left[dr_t^{oe} - r_t^f \, dt \right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i}) \tilde{\sigma}^2 + (1 - \eta_t) \delta^b. \tag{95}$$

It follows that

$$\lambda_t^i = 0, \tag{96}$$

which implies χ_t can be any value in $[\underline{\chi}, 1]$.

Equilibrium refinement To determine the amount of outside equity issuance, we perturb the bubble equilibrium by allowing "trembling hands" of agents. Assume that there is $\epsilon > 0$ chance that agents play for the fundamental equilibrium and $1 - \epsilon$ chance for the bubble equilibrium. We have

$$\frac{\mathbb{E}\left[dr_t^{oe} - r_t^f\, dt\right]}{dt} = (1-\epsilon)(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 + (1-\eta_t)\delta^b, \tag{97}$$

which implies

$$\lambda_t^i = \epsilon(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 > 0, \tag{98}$$

and the equity constraint binds, $\chi_t = \underline{\chi}$. The return of capital is as follows

$$\begin{aligned}\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f\, dt\right]}{dt} &= (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 - (1-\underline{\chi})\epsilon(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 + (1-\eta_t)\delta^b \\ &= (1 - \epsilon + \underline{\chi}\epsilon)(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 + (1-\eta_t)\delta^b.\end{aligned} \tag{99}$$

Taking the limit of χ_t as $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} \chi_t = \underline{\chi}. \tag{100}$$

As long as there is a positive possibility of the fundamental equilibrium, we cannot find a sequence of mixed strategies converging to $\chi_t \neq \underline{\chi}$. We focus on the trembling-hand perfect equilibrium where $\chi_t = \underline{\chi}$ for the following analysis. This trembling-hand perfect equilibrium creates the highest value of bubble.

Steady state in bubble equilibrium

In the bubble economy, the total wealth of the economy is $q_t K_t + P_t = q_t K_t + (1-\underline{\chi}) p_t K_t$.

From the entrepreneur's optimization problem, we have the asset pricing equation for capital

$$\frac{\mathbb{E}[dr_t^{k,i} - r_t^f\, dt]}{dt} = \underbrace{\frac{q_t \underline{\chi} \tilde{\sigma}}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(p_t + q_t)]}}_{\text{price of risk}} \underbrace{\tilde{\sigma}}_{\text{risk}} + (1-\eta_t)\delta^b. \tag{101}$$

In equilibrium, we still have

$$dr_t^{mf} = r_t^f\, dt + (1-\eta_t)\delta^b. \tag{102}$$

To determine the value of the bubble, write the return of public equity as

$$\begin{aligned} dr_t^{mf} &= \underbrace{\frac{(1-\underline{\chi})aK_t}{V_t^{mf}}}_{\text{dividend yield}} dt + \underbrace{\frac{dV_t^{mf}}{V_t^{mf}}}_{\text{capital gain}} = \frac{a}{p_t + q_t} dt + \frac{d\big((p_t+q_t)K_t\big)}{(p_t+q_t)K_t} \\ &= \left(\frac{a + p_t \mu_t^p + q_t \mu_t^q}{p_t + q_t} + g\right) dt, \end{aligned} \tag{103}$$

where $1-\underline{\chi}$ is the fraction of capital issued as outside equity, and V_t^{mf} is the value of the mutual fund.

The consumption goods market-clearing condition is

$$aK_t = (\rho^e \eta_t^b + \rho(1-\eta_t^b))(q_t K_t + P_t). \tag{104}$$

Deriving the evolution of entrepreneurs' wealth share in the bubbly economy, we have

$$\frac{d\eta_t^b}{\eta_t^b} = \underbrace{(1-\eta_t^b)\left(-(\delta^b-\delta^e) + \left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(q_t+p_t)]}\right)^2\right)}_{\equiv \mu_t^{\eta,b}} dt. \tag{105}$$

The risk-free rate is given by

$$r_t^f = \rho - \delta^b + \mu_t^{c,s,b} + g = \rho^e + \mu_t^{c,e,b} - \underbrace{\left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(q_t+p_t)]}\right)^2}_{\text{precautionary saving motive}}, \tag{106}$$

where $\mu_t^{c,s,b}$ and $\mu_t^{c,e,b}$ are the growth rates of savers' and entrepreneurs' consumption, respectively. Since in equilibrium we have $C_t^e = \rho^e \eta_t^b (q_t K_t + P_t)$ and $C_t^s = \rho(1-\eta_t^b)(q_t K_t + P_t)$, we have

$$\frac{dc_t^{e,i}}{c_t^{e,i}} = \mu_t^{c,e,b}\, dt + \tilde{\pi}^{c,e,i}\, d\tilde{Z}_t^i = \left(\mu_t^{\eta,b} + \frac{(1-\underline{\chi})p_t\mu_t^p + q_t\mu_t^q}{q_t + (1-\underline{\chi})p_t} + g\right)dt + \tilde{\pi}^{c,e,i}\, d\tilde{Z}_t^i, \tag{107}$$

$$\frac{dc_t^{s,j}}{c_t^{s,j}} = \mu_t^{c,s,b}\, dt = \left(-\delta^b - \frac{\eta_t^b}{(1-\eta_t^b)}\mu_t^{\eta,b} + \frac{(1-\underline{\chi})p_t\mu_t^p + q_t\mu_t^q}{q_t + (1-\underline{\chi})p_t} + g\right)dt. \tag{108}$$

We solve for prices q_t and p_t as functions of the state variable η_t^b. Combining (101), (102), (103), and (104), and using Itô's lemma, we have

$$\frac{a}{q_t} + \mu_t^q - \frac{a + p_t\mu_t^p + q_t\mu_t^q}{p_t + q_t} = \frac{\underline{\chi} q_t (\tilde{\sigma})^2}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(p_t+q_t)]}, \tag{109}$$

$$\mu_t^q = q'(\eta_t^b)\, \eta_t^b\, \mu_t^{\eta,b}, \tag{110}$$

$$\mu_t^p = p'(\eta_t^b)\, \eta_t^b\, \mu_t^{\eta,b}. \tag{111}$$

Closed-form solution to the steady-state system Let parameters $\{\delta^b, \delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a\}$ be given and define $\Delta \equiv \delta^b - \delta^e > 0$. The system

$$\text{(i)} \quad q + (1-\underline{\chi})\, p \;=\; \frac{a}{\rho - \delta^e \eta}, \tag{112}$$

$$\text{(ii)} \quad \sqrt{\Delta} \;=\; \Gamma^b \;=\; \frac{\underline{\chi}\tilde{\sigma}}{\eta} \cdot \frac{q}{q + (1-\underline{\chi})p}, \tag{113}$$

$$\text{(iii)} \quad \frac{a}{p+q} \;=\; \rho \;-\; \eta\,\delta^b \tag{114}$$

admits the following closed-form solution.

Define the coefficients

$$A \equiv \frac{\delta^b \sqrt{\Delta}}{\tilde{\sigma}}, \qquad B \equiv -\delta^b \;-\; \rho\,\frac{\sqrt{\Delta}}{\tilde{\sigma}} \;+\; (1-\underline{\chi})\,\delta^e, \qquad C \equiv \underline{\chi}\,\rho,$$

and the discriminant $D \equiv B^2 - 4AC$. Then η solves

$$A\,\eta^2 + B\,\eta + C = 0 \quad \Longrightarrow \quad \eta \in \left\{ \frac{-B \pm \sqrt{D}}{2A} \right\},$$

and the economically relevant root is the one that yields $0 < \eta < 1$ and $0 < \frac{\eta\sqrt{\Delta}}{\tilde{\sigma}\underline{\chi}} < 1$.

Given η, the remaining variables are

$$q \;=\; \frac{a\,\eta\,\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}\,(\rho - \delta^e \eta)}, \qquad p \;=\; \frac{a}{\rho - \delta^b \eta} \;-\; q.$$

Equivalently, using (ii),

$$\frac{q}{q + (1-\underline{\chi})p} \;=\; \frac{\eta\,\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}}, \quad \text{and} \quad p \;=\; \frac{a}{(1-\underline{\chi})(\rho - \delta^e \eta)} \left(1 - \frac{\eta\,\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}} \right).$$

Special case $\delta^b = 0$ (linear): if $A = 0$, then η is uniquely

$$\eta \;=\; \frac{\underline{\chi}\,\rho}{\rho\,\frac{\sqrt{\Delta}}{\tilde{\sigma}} - (1-\underline{\chi})\,\delta^e},$$

and

$$q \;=\; \frac{a\,\eta\,\sqrt{\Delta}}{\tilde{\sigma}\,\underline{\chi}\,(\rho - \delta^e \eta)}, \qquad p \;=\; \frac{a}{\rho} - q.$$

Algebraic solution for q_t and p_t in terms of η_t Fix $0 < \underline{\chi} < 1$ and let

$$A_t \equiv \rho - \delta^e \eta_t, \qquad \Gamma_t^b \equiv \frac{\tilde{\sigma}\,\underline{\chi}}{\eta_t} \cdot \frac{q_t A_t}{a}.$$

From the accounting identity

$$(\rho - \delta^e \eta_t)\left(q_t + (1-\underline{\chi})p_t\right) = a$$

and the definition of Γ_t^b, q_t and p_t can be written (pointwise in t) as

$$q_t = \frac{a}{A_t}\,\frac{\eta_t\,\Gamma_t^b}{\tilde{\sigma}\,\underline{\chi}}, \tag{115}$$

$$p_t = \frac{1}{1-\underline{\chi}}\left(\frac{a}{A_t} - q_t\right) = \frac{a}{A_t} \cdot \frac{1 - \dfrac{\eta_t\,\Gamma_t^b}{\tilde{\sigma}\,\underline{\chi}}}{1-\underline{\chi}}. \tag{116}$$

Equivalently, the total value and portfolio weight are

$$p_t + q_t = \frac{a}{A_t} \cdot \frac{1 - \dfrac{\eta_t\,\Gamma_t^b}{\tilde{\sigma}}}{1-\underline{\chi}}, \qquad \frac{q_t}{p_t + q_t} = \frac{1-\underline{\chi}}{\underline{\chi}} \cdot \frac{\dfrac{\eta_t\,\Gamma_t^b}{\tilde{\sigma}}}{1 - \dfrac{\eta_t\,\Gamma_t^b}{\tilde{\sigma}}}.$$

Using the η–dynamics

$$\frac{\dot{\eta}_t}{\eta_t} = (1-\eta_t)\left[(\Gamma_t^b)^2 - \Delta\right], \qquad \Delta \equiv \delta^b - \delta^e > 0,$$

one can eliminate Γ_t^b and express q_t, p_t purely in terms of η_t and $\dot{\eta}_t$:

$$\Gamma_t^b = \sqrt{\Delta + \frac{\dot{\eta}_t}{\eta_t(1-\eta_t)}} \quad (\Gamma_t^b \geq 0), \tag{117}$$

$$q_t = \frac{a}{\rho - \delta^e \eta_t} \cdot \frac{\eta_t}{\tilde{\sigma}\,\underline{\chi}} \sqrt{\Delta + \frac{\dot{\eta}_t}{\eta_t(1-\eta_t)}}, \tag{118}$$

$$p_t = \frac{a}{\rho - \delta^e \eta_t} \cdot \frac{1 - \dfrac{\eta_t}{\tilde{\sigma}\,\underline{\chi}}\sqrt{\Delta + \dfrac{\dot{\eta}_t}{\eta_t(1-\eta_t)}}}{1-\underline{\chi}}. \tag{119}$$

Furthermore,

$$\frac{d\eta_t^b}{\eta_t^b} = \mu_t^{\eta,b}\,dt = (1-\eta_t^b)\left[(\Gamma_t^b)^2 - \Delta\right]dt, \qquad \Gamma_t^b \equiv \frac{\tilde{\sigma}\,\underline{\chi}}{\eta_t^b} \cdot \frac{q_t^b A_t}{a},$$

with $A_t = \rho - \delta^e \eta_t^b$ and $\Delta = \delta^b - \delta^e > 0$. For later use, the growth rates implied by the pricing conditions are

$$\mu_t^q \;=\; \Gamma_t^b \tilde{\sigma} - g + r_t^f + (1-\eta_t)\delta^b - \frac{a}{q_t} \;=\; \Gamma_t^b \tilde{\sigma} - g + r_t^f + (1-\eta_t)\delta^b - \frac{\tilde{\sigma}\,\underline{\chi}\,A_t}{\eta_t\,\Gamma_t^b},$$

and μ_t^p is pinned down by

$$\frac{a}{p_t+q_t} + \frac{q_t}{p_t+q_t}\mu_t^q + \frac{p_t}{p_t+q_t}\mu_t^p + g - r_t^f - (1-\eta_t)\delta^b = 0,$$

with p_t, q_t and the weights substituted from above.

Over time, $q_t K_t$ needs to additionally support return $(1-\eta_t) q_t K_t \delta^b \, dt$ due to population growth.

B.4 Public debt

Government budget constraint

The dynamics of the government bond stock are

$$dB_t \;=\; \left[r_t^f B_t \;+\; sK_t \;-\; \tau\Big(\tfrac{a}{q_t} + g + \mu_t^q\Big) q_t K_t \right] dt.$$

Balanced-Growth Steady State. In a balanced-growth steady state with constant ratios (e.g., B_t/K_t), we impose $dB_t = gB_t$, $\mu_t^q = 0$, and the saver's Euler equation for the safe asset

$$r_t^f \;=\; \rho - \delta^b + g,$$

where $\delta^b \geq 0$ captures, in this environment, the birth rate of traditional savers. The steady-state government budget condition is then

$$g\,\bar{B} \;=\; r^f \bar{B} \;+\; sK \;-\; \tau\Big(\tfrac{a}{\bar{q}} + g\Big)\bar{q}K,$$

i.e.

$$(r^f - g)\,\bar{B} \;+\; sK \;-\; \tau\big(a + g\bar{q}\big)K \;=\; 0.$$

Using $r^f - g = \rho - \delta^b$ gives

$$(\rho - \delta^b)\,\bar{B} \;+\; sK \;-\; \tau\big(a + g\bar{q}\big)K \;=\; 0,$$

so

$$\bar{B} \;=\; \frac{\tau\big(a + g\,\bar{q}\big)K - sK}{\rho - \delta^b}.$$

It is convenient to work with the debt-to-capital ratio $\bar{b} \equiv \bar{B}/K$:

$$\bar{b} \;=\; \frac{\tau\big(a + g\,\bar{q}\big) - s}{\rho - \delta^b}.$$

Goods market clearing

Define $\rho^e \equiv \rho - \delta^e$ with $\delta^e > 0$. Goods market equilibrium satisfies

$$\rho^e \eta_t \big(q_t K_t + B_t\big) \;+\; \rho(1-\eta_t)\big(q_t K_t + B_t\big) \;+\; sK_t \;=\; aK_t.$$

In steady state:

$$\big[\,\rho^e \bar{\eta} + \rho(1-\bar{\eta})\,\big]\big(\bar{q}K + \bar{B}\big) \;=\; (a-s)K, \qquad \bar{q}K + \bar{B} \;=\; \frac{(a-s)K}{\rho^e \bar{\eta} + \rho(1-\bar{\eta})}.$$

Equivalently, in ratio form with $\bar{b} = \bar{B}/K$:

$$\bar{q} + \bar{b} \;=\; \frac{a-s}{\rho^e \bar{\eta} + \rho(1-\bar{\eta})}.$$

Wealth evolution

The entrepreneurs' wealth share evolves according to

$$d\eta_t \;=\; \eta_t(1-\eta_t)\left[-\delta^b \;+\; \delta^e \;+\; \left(\frac{q_t K_t\,(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta_t\big(q_t K_t + B_t\big)}\right)^2\right] dt,$$

where δ^b is the birth rate of traditional savers and we assume $\delta^b > \delta^e > 0$. In steady state ($d\eta_t = 0$):

$$0 \;=\; -\delta^b + \delta^e + \left(\frac{\bar{q}K\,(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\bar{\eta}\,(\bar{q}K + \bar{B})}\right)^2,$$

which implies

$$\frac{\bar{q}K\,(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\bar{\eta}\,(\bar{q}K + \bar{B})} \;=\; \sqrt{\delta^b - \delta^e}, \qquad \bar{\eta} \;=\; \frac{\bar{q}\,(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\sqrt{\delta^b - \delta^e}\,(\bar{q} + \bar{b})}.$$

Individual tax

The tax paid by an individual entrepreneur is

$$\text{Tax}_i \;=\; \tau\Big(\tfrac{a}{q_t} + g + \mu_t^q\Big) q_t k_t^i \, dt.$$

Aggregating across firms (equivalently, across entrepreneurs' inside positions):

$$\text{Total Tax Revenue} \;=\; \tau\Big(\tfrac{a}{q_t} + g + \mu_t^q\Big) q_t K_t.$$

Steady-state solutions

The balanced-growth steady state imposes $dB = gB$, $\mu^q = 0$, and $r^f = \rho - \delta^b + g$. The steady-state values solve

$$\bar{b} = \frac{\tau(a + g\bar{q}) - s}{\rho - \delta^b}, \qquad \bar{q} + \bar{b} = \frac{a - s}{\rho^e \bar{\eta} + \rho(1 - \bar{\eta})},$$

$$\bar{\eta} = \frac{\bar{q}\,(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\sqrt{\delta^b - \delta^e}\,(\bar{q} + \bar{b})}.$$

Equivalently, in levels:

$$\bar{B} = \frac{\tau\big(a + g\,\bar{q}\big)K - sK}{\rho - \delta^b}, \qquad \bar{q}K + \bar{B} = \frac{(a - s)K}{\rho^e \bar{\eta} + \rho(1 - \bar{\eta})}.$$

This is a system in $(\bar{q}, \bar{b}, \bar{\eta})$; once solved, $\bar{B} = \bar{b}K$ and $r^f = \rho - \delta^b + g$.

Existence and admissibility. A feasible solution requires $a > s$, $0 < \bar{\eta} < 1$, and $\rho - \delta^b > 0$ for a well-defined $\bar{b}$. For $\bar{\eta}$ to be well-defined from the wealth-share condition, require $\delta^b > \delta^e$.

B.5 Deriving the Welfare Equation

Entrepreneur's optimization problem

The entrepreneur's problem is to maximize expected utility:

$$V^{e,i,f}(W_t^{e,i}) = \max_{\{c_t^{e,i}, \theta_t^{k,i}, \theta_t^{oe,i}, \theta_t^{mf,i}\}} \mathbb{E}\left[\int_t^\infty e^{-\rho^e(s-t)} \log c_s^{e,i}\, ds \,\middle|\, W_t^{e,i}\right]$$

subject to the wealth dynamics:

$$\frac{dW_t^{e,i}}{W_t^{e,i}} = r_t^f\, dt + \theta_t^{k,i}\mathbb{E}\left[dr_t^{k,i} - r_t^f\, dt\right] + \theta_t^{oe,i}\mathbb{E}\left[dr_t^{oe,i} - r_t^f\, dt\right] + \theta_t^{mf,i}\mathbb{E}\left[dr_t^{mf} - r_t^f\, dt\right]$$
$$- \frac{c_t^{e,i}}{W_t^{e,i}} dt - (1 - \eta_t)\big(\theta_t^{k,i} + \theta_t^{oe,i} + \theta_t^{mf,i}\big)\delta^b\, dt + \left(\theta_t^{k,i} + \theta_t^{oe,i}\right)\tilde{\sigma}\, d\tilde{Z}_t^i,$$

and the equity constraint:

$$-\theta_t^{oe,i} \leq (1 - \underline{\chi})\theta_t^{k,i}.$$

Given $\theta_t^{k,i} = \frac{1}{\eta_t^f} \cdot \frac{q_t^f}{q_t^f + b_t}$, and assuming the equity constraint binds in the fundamental equilibrium, we have:

$$\theta_t^{k,i} + \theta_t^{oe,i} = \underline{\chi}\theta_t^{k,i}, \qquad \theta_t^{oe,i} = (\underline{\chi} - 1)\theta_t^{k,i} = (\underline{\chi} - 1)\frac{1}{\eta_t^f} \cdot \frac{q_t^f}{q_t^f + b_t}.$$

Consumption choice

With log utility and discount rate $\rho^e = \rho - \delta^e$,

$$c_t^{e,i} \;=\; \rho^e\, W_t^{e,i} \;=\; (\rho - \delta^e)\, W_t^{e,i}.$$

Wealth dynamics

The pre-tax return on private equity is

$$dr_t^{k,i} \;=\; \left(\frac{a}{q_t^f} + g + \mu_t^q\right)dt \;+\; \tilde{\sigma}\, d\tilde{Z}_t^i.$$

A proportional tax $\tau \in [0, 1)$ is levied on the whole return to capital, so the after-tax risky excess exposure is scaled by $(1 - \tau)$. Under the binding constraint, the expected after-tax excess return per unit of inside equity satisfies

$$\frac{\mathbb{E}\Big[(1-\tau)dr_t^{k,i} - r_t^f\, dt\Big]}{dt} \;=\; \frac{1}{\eta_t^f}\left((1-\tau)\,\underline{\chi}\,\tilde{\sigma}\cdot\frac{q_t^f}{q_t^f + b_t}\right)^2 + (1-\eta_t)\delta^b.$$

Outside equity loads on the same idiosyncratic shock but earns zero expected excess return in the fundamental equilibrium:

$$dr_t^{oe,i} \;=\; \mathbb{E}[dr_t^{oe,i}] \;+\; \tilde{\sigma}\, d\tilde{Z}_t^i, \qquad \frac{\mathbb{E}\big[dr_t^{oe,i}\big]}{dt} \;=\; r_t^f + (1-\eta_t)\delta^b.$$

Public equity is also free from idiosyncratic risks: $dr_t^{mf} - (1-\eta_t)\delta^b = r_t^f\, dt$.

The diffusion in wealth follows from the binding constraint:

$$\big(\theta_t^{k,i} + \theta_t^{oe,i}\big)(1-\tau)\tilde{\sigma}\, d\tilde{Z}_t^i \;=\; \underline{\chi}\,\theta_t^{k,i}\,(1-\tau)\tilde{\sigma}\, d\tilde{Z}_t^i \;=\; \underline{\chi}\,\frac{(1-\tau)\tilde{\sigma}}{\eta_t^f}\, d\tilde{Z}_t^i.$$

The drift of wealth is

$$\mu_t^{w,e,i} = r_t^f + \left(\frac{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta_t^f}\cdot\frac{q_t^f}{q_t^f + b_t}\right)^2 - (\rho - \delta^e).$$

Hence the individual wealth process is

$$\frac{dW_t^{e,i}}{W_t^{e,i}} = \left[r_t^f + \left(\frac{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta_t^f}\cdot\frac{q_t^f}{q_t^f + b_t}\right)^2 - (\rho - \delta^e)\right]dt \;+\; \underline{\chi}\,\frac{(1-\tau)\tilde{\sigma}}{\eta_t^f}\, d\tilde{Z}_t^i.$$

Wealth-share dynamics (aggregate)

Consistent with the aggregate accounting (and the convention $\delta^b > \delta^e > 0$), the entrepreneurs' wealth share obeys

$$d\eta_t^f \;=\; \eta_t^f(1-\eta_t^f)\left[-\delta^b + \delta^e + \left(\frac{q_t^f\,(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta_t^f\,(q_t^f + b_t)}\right)^2\right] dt.$$

Price and risk-free rate identities

Capital price as a function of the distribution (goods clearing):

$$q_t^f \;=\; \frac{a-s}{\rho^e\eta_t^f + \rho(1-\eta_t^f)} - b_t \;=\; \frac{a-s}{\rho-\delta^e\eta_t^f} - b_t.$$

Total private wealth (valuation):

$$V(\eta_t) = V_t = q_t^f + b_t = \frac{a-s}{\rho-\delta^e\eta_t^f}. \tag{120}$$

Its drift:

$$\mu_t^V \;=\; \frac{dV_t}{V_t} \;=\; \frac{\delta^e\eta_t^f}{\rho-\delta^e\eta_t^f}\,\mu_t^{\eta,f}.$$

Risk-free rate:

$$r_t^f \;=\; \rho-\delta^b+g \;-\; \frac{\eta_t^f}{1-\eta_t^f}\,\mu_t^{\eta,f} \;+\; \mu_t^V = \rho-\delta^b+g \;-\; \frac{(\rho-\delta^e)\eta_t^f}{(1-\eta_t^f)(\rho-\delta^e\eta_t^f)}\,\mu_t^{\eta,f}.$$

Value functions

The value of an entrepreneur consuming $c_t^{e,i} = \rho^e W_t^{e,i}$ is

$$V^{e,i,f}(W_t^{e,i}) = \mathbb{E}\left[\int_t^\infty e^{-\rho^e(s-t)}\log\left(\rho^e W_s^{e,i}\right)ds\,\middle|\,W_t^{e,i}\right] \tag{121}$$

$$= \frac{1}{\rho^e}\log\rho^e + \frac{1}{\rho^e}\log W_t^{e,i} + \frac{1}{\rho^e}\int_0^\infty e^{-\rho^e u}\,\Psi^e(t+u)\,du, \tag{122}$$

where

$$\Psi^e(t) = r_t^f + \left(\frac{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta_t^f}\cdot\frac{q_t^f}{q_t^f+b_t}\right)^2 - (\rho-\delta^e) - \frac{1}{2}\left(\frac{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta_t^f}\cdot\frac{q_t^f}{q_t^f+b_t}\right)^2.$$

Using the identity for r_t^f above, this simplifies to

$$\Psi^e(t) = g - (\delta^b - \delta^e) - \frac{\eta_t^f}{1-\eta_t^f}\mu_t^{\eta,f} + \mu_t^V + \frac{1}{2}\left(\frac{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta_t^f} \cdot \frac{q_t^f}{q_t^f + b_t}\right)^2$$
$$= g - \frac{\rho(1-\eta_t^f)}{\rho - \delta^e \eta_t^f}(\delta^b - \delta^e) + \left(\frac{1}{2} - \frac{(\rho - \delta^e)\eta_t^f}{\rho - \delta^e \eta_t^f}\right)\left(\frac{(1-\tau)\,\underline{\chi}\,\tilde{\sigma}}{\eta_t^f} \cdot \frac{q_t^f}{q_t^f + b_t}\right)^2.$$

Savers' value is analogous with ρ and without the variance term.

B.6 Global Welfare in Fundamental and Bubble Equilibria, and Policy Effects

Preliminaries and common objects

Parameters: $(\rho, \delta^e, \delta^b, a, g, \underline{\chi}, \tilde{\sigma})$ with $\delta^b > \delta^e > 0$ and $\rho > \delta^e$. Define $\rho^e := \rho - \delta^e > 0$, $\Delta := \delta^b - \delta^e > 0$, state $\eta_t \in (0,1)$, and $A_t := \rho - \delta^e \eta_t > 0$.

Per-unit total public-equity value:

$$V(\eta) := \frac{a}{\rho - \delta^e \eta} = \frac{a}{A(\eta)}.$$

Safe-rate identity (valid in both regimes):

$$r_t^f = \rho - \delta^b + g + B(\eta_t)\,\mu_t^\eta, \qquad B(\eta) \equiv \frac{\delta^e \eta}{\rho - \delta^e \eta} - \frac{\eta}{1-\eta}, \qquad \mu_t^\eta := \frac{1}{\eta_t}\frac{d\eta_t}{dt}. \tag{123}$$

If $\rho > \delta^e$ and $\eta \in (0,1)$ then $B(\eta) < 0$.

Fundamental equilibrium (f): policies, dynamics, welfare

Issuance constraint binds at $\chi_t = \underline{\chi}$. Define exposure per unit wealth

$$\Gamma_t^f = \frac{\underline{\chi}\tilde{\sigma}}{\eta_t}.$$

Log consumption rule: $c_t^{e,i} = \rho^e W_t^{e,i}$. Individual wealth SDE:

$$\frac{dW_t^{e,i}}{W_t^{e,i}} = \left[r_t^f - \rho^e + (\Gamma_t^f)^2\right]dt + \Gamma_t^f\, d\tilde{Z}_t^i. \tag{124}$$

Aggregate distribution:

$$\mu_t^{\eta,f} = (1-\eta_t)\left[-\Delta + \left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t}\right)^2\right]. \tag{125}$$

Entrepreneurs' instant surplus:

$$\Psi^{e,f}(t) = [r_t^f - \rho^e] + \tfrac{1}{2}(\Gamma_t^f)^2 = \left[\rho - \delta^b + g - \rho^e\right] + B(\eta_t)\,\mu_t^{\eta,f} + \tfrac{1}{2}\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t}\right)^2.$$

Savers' instant surplus:

$$\Psi^{s,f}(t) = r_t^f - \rho = (g - \delta^b) + B(\eta_t)\,\mu_t^{\eta,f}.$$

Steady state: $\bar{\eta}^f = \underline{\chi}\tilde{\sigma}/\sqrt{\Delta}$, $\bar{q}^f = a/(\rho - \delta^e\bar{\eta}^f)$, $\bar{r}^f = \rho - \delta^b + g$.

Bubble equilibrium (b): composition, dynamics, welfare

Composition identity:

$$q_t^b + (1-\underline{\chi})p_t = V(\eta_t) = \frac{a}{A_t}.$$

Exposure:

$$\Gamma_t^b = \frac{\underline{\chi}\tilde{\sigma}}{\eta_t} \cdot \frac{q_t^b A_t}{a} \in \left(0, \frac{\underline{\chi}\tilde{\sigma}}{\eta_t}\right].$$

Aggregate distribution:

$$\mu_t^{\eta,b} = (1-\eta_t)\left[-\Delta + (\Gamma_t^b)^2\right]. \tag{126}$$

Safe rate: $r_t^f = \rho - \delta^b + g + B(\eta_t)\,\mu_t^{\eta,b}$. Entrepreneurs' and savers' values are as above with $(\cdot)^f \to (\cdot)^b$.

Any interior b steady state with $p > 0$ satisfies $(\Gamma^b)^2 = \Delta$, implying $\bar{r}^f = \rho - \delta^b + g$ and $\bar{\eta}^b < \bar{\eta}^f$.

Welfare comparison: f vs b (pointwise at matched η)

Let $C(\eta) \equiv \frac{1}{2} - \frac{(\rho - \delta^e)\eta}{\rho - \delta^e\eta}$. Then

$$\Psi^{e,b} - \Psi^{e,f} = C(\eta)\left[(\Gamma^b)^2 - (\Gamma^f)^2\right], \qquad \Psi^{s,b} - \Psi^{s,f} = B(\eta)\left(\mu^{\eta,b} - \mu^{\eta,f}\right) \geq 0.$$

Public debt and taxes: exposures, rates, welfare (pointwise)

With bonds b_t and tax τ:

$$q_t^b + (1-\underline{\chi})\,p_t + b_t \;=\; V(\eta_t), \qquad \theta_t \equiv \frac{q_t^b}{V(\eta_t)}.$$

Exposures in regime $r \in \{f, b\}$:

$$\Gamma_t^{r,\tau} = (1-\tau)\,\frac{\underline{\chi}\,\tilde{\sigma}}{\eta_t} \times \begin{cases} \theta_t, & r = b, \\ 1 - \dfrac{b_t}{V(\eta_t)}, & r = f \text{ (since } p_t = 0). \end{cases}$$

Wealth-share drift and safe rate:

$$\mu_t^{\eta,r,\tau} = (1-\eta_t)\big[(\Gamma_t^{r,\tau})^2 - \Delta\big], \qquad r_t^{f,r,\tau} = \rho - \delta^b + g + B(\eta_t)\,\mu_t^{\eta,r,\tau}.$$

Pointwise changes at matched η:

$$\Delta\Psi^e = C(\eta)\,\Delta(\Gamma^2), \qquad \Delta\Psi^s = B(\eta)(1-\eta)\,\Delta(\Gamma^2).$$

Thus safer policies ($\Delta(\Gamma^2) < 0$) raise savers' flow welfare; entrepreneurs' effect flips sign at $\eta = \rho/(2\rho - \delta^e)$.

Bubble region with $p_t > 0$: a small db_t crowds out $(1-\underline{\chi})p_t$ one-for-one with no first-order effect on q_t^b, Γ, r^f, or pointwise welfare.

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